Notes for Lecture #3

Form of Green’s function solutions to the Poisson equation

According to Eq. 1.35 of your text for any two three-dimensional functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$
\int_{\text{Vol}} \left( \phi(\mathbf{r}) \nabla^2 \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) \right) d^3r = \oint_{\text{Surf}} \left( \phi(\mathbf{r}) \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla \phi(\mathbf{r}) \right) \cdot \hat{n} d^2r,
$$

(1)

where $\hat{n}$ denotes a unit vector normal to the integration surface. We can choose to evaluate this expression with $\phi(\mathbf{r}) = \Phi(\mathbf{r})$ (the electrostatic potential) and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$, and also make use of the identities:

$$
\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}
$$

(2)

and

$$
\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}').
$$

(3)

Then, the Green’s identity (1) becomes

$$
-4\pi \int_{\text{Vol}} \left( \Phi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r})}{4\pi \varepsilon_0} \right) d^3r = \oint_{\text{Surf}} \left\{ \Phi(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \nabla \Phi(\mathbf{r}) \right\} \cdot \hat{n} d^2r.
$$

(4)

This expression can be further evaluated. If the arbitrary position, $\mathbf{r}'$ is included in the integration volume, then the equation (4) becomes

$$
\Phi(\mathbf{r}') = \int_{\text{Vol}} G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r})}{4\pi \varepsilon_0} d^3r + \frac{1}{4\pi} \oint_{\text{Surf}} \left\{ G(\mathbf{r}, \mathbf{r}') \nabla \Phi(\mathbf{r}) - \Phi(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') \right\} \cdot \hat{n} d^2r.
$$

(5)

This expression is the same as Eq. 1.42 of your text if we switch the variables $\mathbf{r}' \leftrightarrow \mathbf{r}$ and also use the fact that Green’s function is symmetric in its arguments: $G(\mathbf{r}, \mathbf{r}') \equiv G(\mathbf{r}', \mathbf{r})$.

Mean value theorem for solutions to the Laplace equation

Consider an electrostatic field $\Phi(\mathbf{r})$ in a charge-free region so that it satisfies the Laplace equation:

$$
\nabla^2 \Phi(\mathbf{r}) = 0.
$$

(6)

The “mean value theorem” value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the
surface of any sphere centered on the point \( \mathbf{r} \) (see Jackson problem \#1.10). One way to prove this theorem is the following. Consider a point \( \mathbf{r}' = \mathbf{r} + \mathbf{u} \), where \( \mathbf{u} \) will describe a sphere of radius \( R \) about the fixed point \( \mathbf{r} \). We can make a Taylor series expansion of the electrostatic potential \( \Phi(\mathbf{r}') \) about the fixed point \( \mathbf{r} \):

\[
\Phi(\mathbf{r} + \mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u} \cdot \nabla \Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u} \cdot \nabla)^2 \Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u} \cdot \nabla)^3 \Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u} \cdot \nabla)^4 \Phi(\mathbf{r}) + \cdots. \tag{7}
\]

According to the premise of the theorem, we want to integrate both sides of the equation 7 over a sphere of radius \( R \) in the variable \( \mathbf{u} \):

\[
\int_{\text{sphere}} dS_u = R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \tag{8}
\]

We note that

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) = 4\pi R^2, \tag{9}
\]

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u} \cdot \nabla = 0, \tag{10}
\]

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^2 = \frac{4\pi R^4}{3} \nabla^2, \tag{11}
\]

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^3 = 0, \tag{12}
\]

and

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^4 = \frac{4\pi R^6}{5} \nabla^4. \tag{13}
\]

Since \( \nabla^2 \Phi(\mathbf{r}) = 0 \), the only non-zero term of the average it thus the first term:

\[
R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}), \tag{14}
\]

or

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} R^2 \int_{0}^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}). \tag{15}
\]

Since this result is independent of the radius \( R \), we see that we have proven the theorem.