

Notes for Lecture #12

Dipole and quadrupole fields

The dipole moment is defined by

$$\mathbf{p} = \int d^3r \rho(r) \mathbf{r}, \quad (1)$$

with the corresponding potential

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}, \quad (2)$$

and electrostatic field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{3\hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}}) - \mathbf{p}}{r^3} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right\}. \quad (3)$$

The last term of the field expression follows from the following derivation. We note that Eq. (3) is poorly defined as $r \rightarrow 0$, and consider the value of a small integral of $\mathbf{E}(\mathbf{r})$ about zero. (For this purpose, we are supposing that the dipole \mathbf{p} is located at $\mathbf{r} = \mathbf{0}$.) In this case we will approximate

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3\mathbf{r} \right) \delta^3(\mathbf{r}). \quad (4)$$

First we note that

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3\mathbf{r} = -R^2 \int_{\mathbf{r}=\mathbf{R}} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega. \quad (5)$$

This result follows from the Divergence theorem:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} d^3\mathbf{r} = \int_{\text{surface}} \mathcal{V} \cdot d\mathbf{A}. \quad (6)$$

In our case, this theorem can be used to prove Eq. (5) for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the x - component for example:

$$\int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3\mathbf{r} = \hat{\mathbf{x}} \int_{\mathbf{r} \leq R} \nabla \cdot (\hat{\mathbf{x}}\Phi) d^3\mathbf{r} + \hat{\mathbf{y}} \int_{\mathbf{r} \leq R} \nabla \cdot (\hat{\mathbf{y}}\Phi) d^3\mathbf{r} + \hat{\mathbf{z}} \int_{\mathbf{r} \leq R} \nabla \cdot (\hat{\mathbf{z}}\Phi) d^3\mathbf{r}, \quad (7)$$

which is equal to

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega ((\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})\hat{\mathbf{z}}) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega \hat{\mathbf{r}}. \quad (8)$$

Thus,

$$\int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3\mathbf{r} = - \int_{\mathbf{r} \leq R} \nabla \Phi(\mathbf{r}) d^3\mathbf{r} = -R^2 \int_{\mathbf{r}=\mathbf{R}} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega. \quad (9)$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \int d^3r' \rho(\mathbf{r}') \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}'). \quad (10)$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta) \cos(\phi) \hat{\mathbf{x}} + \sin(\theta) \sin(\phi) \hat{\mathbf{y}} + \cos(\theta) \hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}) \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}) \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}) \hat{\mathbf{z}} \right) \end{cases} \quad (11)$$

Therefore, when we evaluate the integral over solid angle Ω in Eq. (5), only the $l = 1$ term contributes and the effect of the integration reduced to the expression:

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \int d^3r' \rho(\mathbf{r}') \frac{\mathbf{r}_{<} - \mathbf{r}_{>}}{r_{>}^2} \hat{\mathbf{r}}'. \quad (12)$$

The choice of $r_{<}$ and $r_{>}$ is a choice between the integration variable r' and the sphere radius R . If the sphere encloses the charge distribution $\rho(\mathbf{r}')$, then $r_{<} = r'$ and $r_{>} = R$ so that Eq. (12) becomes

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} \frac{1}{R^2} \int d^3r' \rho(\mathbf{r}') \mathbf{r}' \hat{\mathbf{r}}' \equiv -\frac{\mathbf{p}}{3\epsilon_0}. \quad (13)$$

If the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_{<} = R$ and $r_{>} = r'$ so that Eq. (12) becomes

$$-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\epsilon_0} \frac{4\pi R^2}{3} R \int d^3r' \frac{\rho(\mathbf{r}')}{r'^2} \hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3} \mathbf{E}(\mathbf{0}), \quad (14)$$

which is consistent with the mean value theorem for the electrostatic potential.