

Notes for Lecture #2

Examples of solutions of the one-dimensional Poisson equation

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \quad (1)$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}, \quad (2)$$

with the boundary condition $\Phi(-\infty) = 0$.

The solution to the differential equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + (\rho_0 a^2)/\varepsilon_0 & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases} \quad (3)$$

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \quad (4)$$

The electrostatic potential can be determined by piecewise solution within each of the four regions or by use of the Green's function $G(x, x') = 4\pi x_{<}$, where,

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} G(x, x')\rho(x')dx'. \quad (5)$$

In the expression for $G(x, x')$, $x_{<}$ should be taken as the smaller of x and x' . It can be shown that Eq. 5 gives the identical result for $\Phi(x)$ as given in Eq. 3.

Notes on the one-dimensional Green's functions

The Green's function for the Poisson equation can be defined as a solution to the equation:

$$\nabla^2 G(x, x') = -4\pi\delta(x - x'). \quad (6)$$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x . It is easily shown that with this definition of the Green's function (6), Eq. (5) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green's function which satisfies Eq. (6), we notice that we can use two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0, \quad (7)$$

where $i = 1$ or 2 , to form

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_<) \phi_2(x_>). \quad (8)$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger. In this expression W is the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx} \phi_2(x) - \phi_1(x) \frac{d\phi_2(x)}{dx}. \quad (9)$$

We can check that this "recipe" works by noting that for $x \neq x'$, Eq. (8) satisfies the defining equation 6 by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 7. The defining equation is singular at $x = x'$, but integrating 6 over x in the neighborhood of x' ($x' - \epsilon < x < x' + \epsilon$), gives the result:

$$\left. \frac{dG(x, x')}{dx} \right|_{x=x'+\epsilon} - \left. \frac{dG(x, x')}{dx} \right|_{x=x'-\epsilon} = -4\pi. \quad (10)$$

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green's function is as given above. For this piecewise continuous form of the Green's function, the integration 5 can be evaluated:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \left\{ \int_{-\infty}^x G(x, x') \rho(x') dx' + \int_x^{\infty} G(x, x') \rho(x') dx' \right\}, \quad (11)$$

which becomes

$$\Phi(x) = \frac{1}{\epsilon_0} \left\{ \int_{-\infty}^x x' \rho(x') dx' + x \int_x^{\infty} \rho(x') dx' \right\}. \quad (12)$$

Evaluating this expression, we find that we obtain the same result as given in Eq. (3).

In general, the Green's function $G(x, x')$ solution (5) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all *neutral* charge densities, that is $\int_{-\infty}^{\infty} \rho(x) dx = 0$.