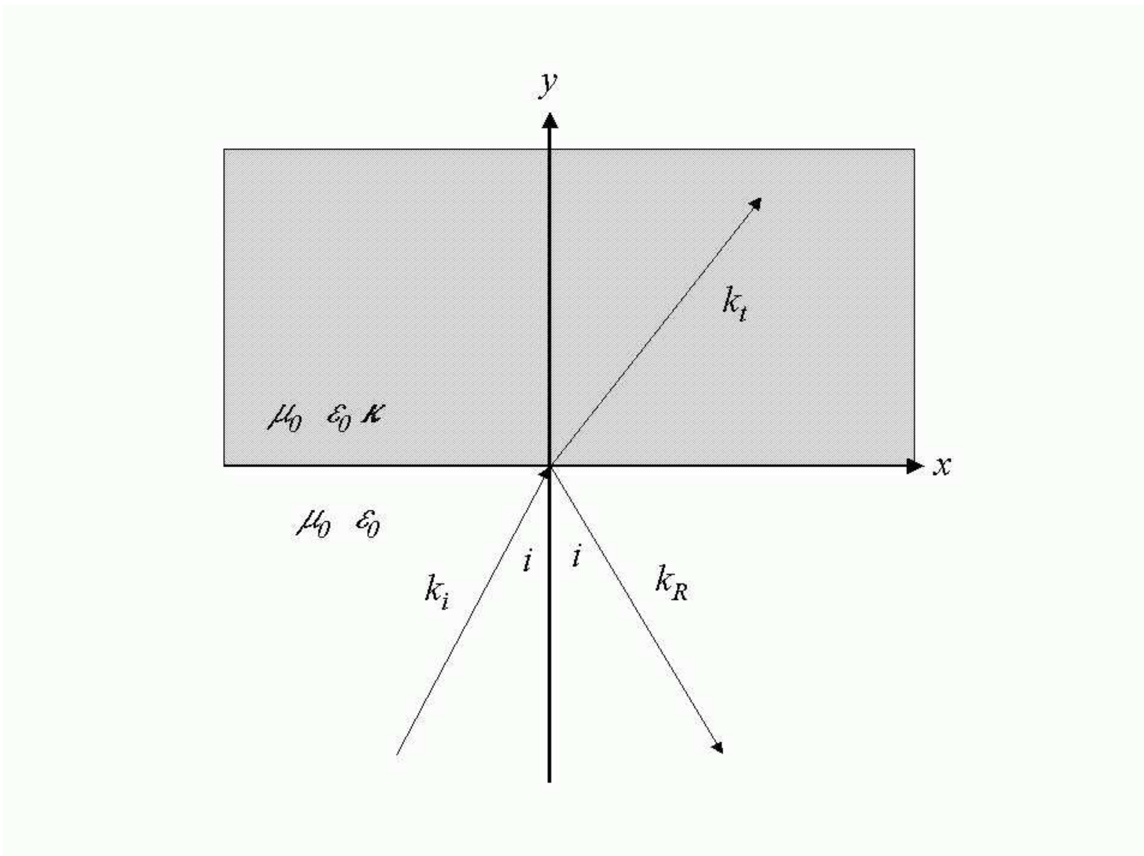
March 29, 2004

Notes for Lecture #23

Reflectivity for anisotropic media – Extension of Section 7.3 in Jackson's text



Consider the problem of determining the reflectance from an isotropic medium as shown above. For simplicity, we will assume that the dielectric tensor for the medium is diagonal and is given by:

$$\kappa \equiv \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}. \tag{1}$$

We will assume also that the wave vector in the medium is confined to the $x-y$ plane and will be denoted by

$$\mathbf{k}_t \equiv \frac{\omega}{c}(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}}). \tag{2}$$

We will assume that the complex representation of electric field inside the medium is given by

$$\mathbf{E} = (E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}})\mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}. \tag{3}$$

In terms of this electric field and the magnetic field $\mathbf{H} = \mathbf{B}/\mu_0$, where $\mathbf{H}$ is assumed to have the same complex spatial and temporal form as (3), the four Maxwell's equations are given by:

$$\begin{aligned} \nabla \cdot \mathbf{H} = 0 \qquad & \qquad \nabla \cdot \kappa \cdot \mathbf{E} = 0 \\ \nabla \times \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \qquad & \qquad \nabla \times \mathbf{H} + i\omega\epsilon_0 \kappa \cdot \mathbf{E} = 0 \end{aligned} \tag{4}$$

Using these equations, we obtain the following equations for electric field amplitudes within the medium:

$$\begin{pmatrix} \kappa_{xx} - n_y^2 & n_x n_y & 0 \\ n_x n_y & \kappa_{yy} - n_x^2 & 0 \\ 0 & 0 & \kappa_{zz} - (n_x^2 + n_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{5}$$

Once the electric field amplitudes are determined, the magnetic field can be determined according to:

$$\mathbf{H} = \frac{1}{\mu_0 c} \left\{ E_z (n_y \hat{\mathbf{x}} - n_x \hat{\mathbf{y}}) + (E_y n_x - E_x n_y) \hat{\mathbf{z}} \right\} \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}. \tag{6}$$

The incident and reflected electro-magnetic fields are given in your textbook. In the notation of the figure the wavevector for the incident wave is given by:

$$\mathbf{k}_i = \frac{\omega}{c} (\sin i \hat{\mathbf{x}} + \cos i \hat{\mathbf{y}}), \tag{7}$$

and the wavevector for reflected wave is given by:

$$\mathbf{k}_R = \frac{\omega}{c} (\sin i \hat{\mathbf{x}} - \cos i \hat{\mathbf{y}}). \tag{8}$$

In this notation, Snell's law requires that $n_x = \sin i$. The continuity conditions at the $y = 0$ plane involve continuity requirements on the following fields:

$$\mathbf{H}(x, 0, z, t), \quad E_x(x, 0, z, t), \quad E_z(x, 0, z, t), \quad \text{and} \quad D_y(x, 0, z, t), \tag{9}$$

at all times t.

Below we consider two different polarizations for the electric field.

Solution for s-polarization

In this case, $E_x = E_y = 0$, and $n_y^2 = \kappa_{zz} - n_x^2$. The fields in the medium are given by:

$$\mathbf{E} = E_z \hat{\mathbf{z}} \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t} \qquad \mathbf{H} = \frac{1}{\mu_0 c} \left\{ E_z (n_y \hat{\mathbf{x}} - n_x \hat{\mathbf{y}}) \right\} \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}. \tag{10}$$

The amplitude E_z can be determined from the matching conditions:

$$\begin{aligned} E_0 + E_0'' &= E_z \\ (E_0 - E_0'') \cos i &= E_z n_y \\ (E_0 + E_0'') \sin i &= E_z n_x. \end{aligned} \tag{11}$$

In this case, the last equation is redundant. The other two equations can be solved for the reflected amplitude:

$$\frac{E_0''}{E_0} = \frac{\cos i - n_y}{\cos i + n_y}. \tag{12}$$

This is very similar to the result given in Eq. 7.39 of **Jackson** for the isotropic media.

Solution for p-polarization

In this case, $E_z = 0$ and

$$n_y^2 = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - n_x^2). \tag{13}$$

In terms of the unknown amplitude E_x, the electric field in the medium is given by:

$$\mathbf{E} = E_x \left(\hat{\mathbf{x}} - \frac{\kappa_{xx} n_x}{\kappa_{yy} n_y} \hat{\mathbf{y}} \right) \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}. \tag{14}$$

The corresponding magnetic field is given by:

$$\mathbf{H} = -\frac{E_x}{\mu_0 c} \frac{\kappa_{xx}}{n_y} \hat{\mathbf{z}} \mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}. \tag{15}$$

The amplitude E_x can be determined from the matching conditions:

$$\begin{gathered}(E_0 - E_0'') \cos i = E_x \\ (E_0 + E_0'') = \tfrac{\kappa_{xx}}{n_y} E_x \\ (E_0 + E_0'') \sin i = \tfrac{\kappa_{xx} n_x}{n_y} E_x.\end{gathered} \tag{16}$$

Again, the last equation is redundant, and the solution for the reflected amplitude is given by:

$$\frac{E_0''}{E_0} = \frac{\cos i \kappa_{xx} - n_y}{\cos i \kappa_{xx} + n_y}. \tag{17}$$

This result reduces to Eq. 7.41 in **Jackson** for the isotropic case.