

Notes for Lecture #3

Form of Green's function solutions to the Poisson equation

According to Eq. 1.35 of your text for any two three-dimensional functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$\int_{\text{Vol}} (\phi(\mathbf{r}) \nabla^2 \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla^2 \phi(\mathbf{r})) d^3r = \oint_{\text{Surf}} (\phi(\mathbf{r}) \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla \phi(\mathbf{r})) \cdot \hat{\mathbf{r}} d^2r, \quad (1)$$

where $\hat{\mathbf{r}}$ denotes a unit vector normal to the integration surface. We can choose to evaluate this expression with $\phi(\mathbf{r}) = \Phi(\mathbf{r})$ (the electrostatic potential) and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$, and also make use of the identities:

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \quad (2)$$

and

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}'). \quad (3)$$

Then, the Green's identity (1) becomes

$$-4\pi \int_{\text{Vol}} \left(\Phi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r})}{4\pi\varepsilon_0} \right) d^3r = \oint_{\text{Surf}} \{ \Phi(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \nabla \Phi(\mathbf{r}) \} \cdot \hat{\mathbf{r}} d^2r. \quad (4)$$

This expression can be further evaluated. If the arbitrary position, \mathbf{r}' is included in the integration volume, then the equation (4) becomes

$$\Phi(\mathbf{r}') = \int_{\text{Vol}} G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r})}{4\pi\varepsilon_0} d^3r + \frac{1}{4\pi} \oint_{\text{Surf}} \{ G(\mathbf{r}, \mathbf{r}') \nabla \Phi(\mathbf{r}) - \Phi(\mathbf{r}) \nabla G(\mathbf{r}, \mathbf{r}') \} \cdot \hat{\mathbf{r}} d^2r. \quad (5)$$

This expression is the same as Eq. 1.42 of your text if we switch the variables $\mathbf{r}' \Leftrightarrow \mathbf{r}$ and also use the fact that Green's function is symmetric in its arguments: $G(\mathbf{r}, \mathbf{r}') \equiv G(\mathbf{r}', \mathbf{r})$.

Mean value theorem for solutions to the Laplace equation

Consider an electrostatic field $\Phi(\mathbf{r})$ in a charge-free region so that it satisfies the Laplace equation:

$$\nabla^2 \Phi(\mathbf{r}) = 0. \quad (6)$$

The “mean value theorem” value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point \mathbf{r} is equal to the average of $\Phi(\mathbf{r}')$ over the

surface of any sphere centered on the point \mathbf{r} (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where \mathbf{u} will describe a sphere of radius R about the fixed point \mathbf{r} . We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point \mathbf{r} :

$$\Phi(\mathbf{r} + \mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u} \cdot \nabla \Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u} \cdot \nabla)^2 \Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u} \cdot \nabla)^3 \Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u} \cdot \nabla)^4 \Phi(\mathbf{r}) + \dots \quad (7)$$

According to the premise of the theorem, we want to integrate both sides of the equation 7 over a sphere of radius R in the variable \mathbf{u} :

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \quad (8)$$

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \quad (9)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u} \cdot \nabla = 0, \quad (10)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^2 = \frac{4\pi R^4}{3} \nabla^2, \quad (11)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^3 = 0, \quad (12)$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^4 = \frac{4\pi R^6}{5} \nabla^4. \quad (13)$$

Since $\nabla^2 \Phi(\mathbf{r}) = 0$, the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}), \quad (14)$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}). \quad (15)$$

Since this result is independent of the radius R , we see that we have proven the theorem.