

# Notes for Lecture #31

## Synchrotron Radiation

For this analysis we will use the geometry shown in Fig. 14.9 of **Jackson**. A particle with charge  $q$  is moving in a circular trajectory with radius  $\rho$  and speed  $v$ . Its trajectory as a function of time  $t$  is given by

$$\mathbf{r}_q(t) = \rho \sin(vt/\rho) \hat{\mathbf{x}} + \rho (1 - \cos(vt/\rho)) \hat{\mathbf{y}}. \quad (1)$$

Its velocity as a function time is given by

$$\mathbf{v}_q(t) = v \cos(vt/\rho) \hat{\mathbf{x}} + v \sin(vt/\rho) \hat{\mathbf{y}}. \quad (2)$$

The spectral intensity that we must evaluate is given by the expression (Eq. 14.67 in Jackson)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}} \cdot \mathbf{r}_q(t))} dt \right|^2. \quad (3)$$

After some algebra, this expression can be put into the form

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{4\pi^2 c} \left\{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \right\}, \quad (4)$$

where the amplitude for the light polarized along the  $y$ -axis is given by

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))} \quad (5)$$

and the amplitude for the light polarized perpendicular to  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{r}}$  is given by

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin \theta \cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))}. \quad (6)$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ( $v \approx c(1 - 1/(2\gamma^2))$ ) passing a beam line port. In addition  $\theta \approx 0$  and the relevant integration times  $t$  are close to  $t \approx 0$ . This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency

$$\omega_c \equiv \frac{3c\gamma^3}{2\rho}. \quad (7)$$

The resultant intensity is then given by

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left( \frac{\omega}{\omega_c} \right)^2 (1 + \gamma^2 \theta^2)^2 \left\{ \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{\frac{3}{2}} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{\frac{3}{2}} \right) \right]^2 \right\}. \quad (8)$$

By plotting this expression as a function of  $\omega$ , we see that the intensity is largest near  $\omega \approx \omega_c$ .

The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of  $v/\rho$ . For this case we need to reconsider Eqs. 5 and 6. There is a very convenient Bessel function identity of the form:

$$e^{-ia \sin \alpha} = \sum_{m=-\infty}^{\infty} J_m(a) e^{-ima}. \quad (9)$$

Here  $J_m(a)$  is a Bessel function of integer order  $m$ . In our case  $a = \frac{\omega \rho}{c} \cos \theta$  and  $\alpha = \frac{vt}{\rho}$ . Analyzing the “parallel” component we have

$$C_{\parallel} = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))} = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega \rho}{c} \cos \theta\right) 2\pi \delta\left(\omega - m \frac{v}{\rho}\right). \quad (10)$$

In determining this result, we have used the identity

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m \frac{v}{\rho})t} = 2\pi \delta\left(\omega - m \frac{v}{\rho}\right). \quad (11)$$

Eq. 10 can be simplified to show that

$$C_{\parallel} = 2\pi i \sum_{m=-\infty}^{\infty} J'_m\left(\frac{\omega \rho}{c} \cos \theta\right) \delta\left(\omega - m \frac{v}{\rho}\right), \quad (12)$$

where  $J'_m(a) \equiv \frac{dJ_m(a)}{da}$ . The “perpendicular” component can be analyzed in a similar way, using integration by parts to eliminate the extra  $\cos(vt/\rho)$  term in the argument. The result is

$$C_{\perp} = 2\pi \frac{\tan \theta}{v/c} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega \rho}{c} \cos \theta\right) \delta\left(\omega - m \frac{v}{\rho}\right). \quad (13)$$

In both of these expressions, the sum over  $m$  includes both negative and positive values of  $m$ . However, only the positive values of  $\omega$  and therefore positive values of  $m$  are of interest, and if we needed to use the negative  $m$  values, we could use the identity

$$J_{-m}(a) = (-1)^m J_m(a). \quad (14)$$

Combining these results, we find that the intensity spectrum for this case consists of a series of discrete frequencies which are multiples of  $v/\rho$ .

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{c} \sum_{m=0}^{\infty} \delta\left(\omega - m \frac{v}{\rho}\right) \left\{ \left[ J'_m\left(\frac{\omega \rho}{c} \cos \theta\right) \right]^2 + \frac{\tan^2 \theta}{v^2/c^2} \left[ J_m\left(\frac{\omega \rho}{c} \cos \theta\right) \right]^2 \right\}. \quad (15)$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson’s text. It should have some implications for Astronomical observations, but I have not yet found any references for that.