

Notes for Lecture #1

1 Introduction

1. Textbook and course structure
2. Motivation
3. Chapters I and 1 and Appendix of Jackson
 - (a) Units - SI vs Gaussian
 - (b) Laplace and Poisson Equations
 - (c) Green's Theorem

2 Units - SI vs Gaussian

Coulomb's law has the form:

$$F = K_C \frac{q_1 q_2}{r_{12}^2}. \quad (1)$$

Ampere's law has the form:

$$F = K_A \frac{i_1 i_2}{r_{12}^2} d\mathbf{s}_1 \times d\mathbf{s}_2 \times \hat{\mathbf{r}}_{12}, \quad (2)$$

where the current and charge are related by $i_1 = dq_1/dt$ for all unit systems. The two constants K_C and K_A are related so that their ratio K_C/K_A has the units of $(m/s)^2$ and it is *experimentally* known that in both the SI and CGS (Gaussian) unit systems, it the value $K_C/K_A = c^2$, where c is the speed of light.

The choices for these constants in the SI and Gaussian units are given below:

| | CGS (Gaussian) | SI |
|-------|-----------------|----------------------------|
| K_C | 1 | $\frac{1}{4\pi\epsilon_0}$ |
| K_A | $\frac{1}{c^2}$ | $\frac{\mu_0}{4\pi}$ |

Here, $\frac{\mu_0}{4\pi} \equiv 10^{-7} N/A^2$ and $\frac{1}{4\pi\epsilon_0} = c^2 \cdot 10^{-7} N/A^2 = 8.98755 \times 10^9 N \cdot m^2/C^2$.

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

| Variable | SI | | Gaussian | | SI/Gaussian |
|----------|------|------------------|---------------|--------------------------|-----------------|
| | Unit | Relation | Unit | Relation | |
| length | m | fundamental | cm | fundamental | 100 |
| mass | kg | fundamental | gm | fundamental | 1000 |
| time | s | fundamental | s | fundamental | 1 |
| force | N | $kg \cdot m^2/s$ | $dyne$ | $gm \cdot cm^2/s$ | 10^5 |
| current | A | fundamental | $statampere$ | $statcoulomb/s$ | $\frac{1}{10c}$ |
| charge | C | $A \cdot s$ | $statcoulomb$ | $\sqrt{dyne \cdot cm^2}$ | $\frac{1}{10c}$ |

One advantage of the Gaussian system is that all of the field vectors: \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} , \mathbf{P} , \mathbf{M} have the same dimensions, and in vacuum, $\mathbf{B} = \mathbf{H}$ and $\mathbf{E} = \mathbf{D}$ and the dielectric and permittivity constants ϵ and μ are unitless.

| CGS (Gaussian) | SI |
|---|--|
| $\nabla \cdot \mathbf{D} = 4\pi\rho$ | $\nabla \cdot \mathbf{D} = \rho$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ |
| $\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$ | $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ |
| $\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ |
| $u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ | $u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ |
| $\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$ | $\mathbf{S} = (\mathbf{E} \times \mathbf{H})$ |

“Proof” of the identity (Eq. (1.31))

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'). \quad (3)$$

Noting that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \delta^3(\mathbf{r} - \mathbf{r}') f(\mathbf{r}) = f(\mathbf{r}'), \quad (4)$$

we see that we must show that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) = -4\pi f(\mathbf{r}'). \quad (5)$$

We introduce a small radius a such that:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \lim_{a \rightarrow 0} \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}}. \quad (6)$$

For a fixed value of a ,

$$\nabla^2 \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}} = \frac{-3a^2}{(|\mathbf{r} - \mathbf{r}'|^2 + a^2)^{5/2}}. \quad (7)$$

If the function $f(\mathbf{r})$ is continuous, we can make a Taylor expansion of it about the point $\mathbf{r} = \mathbf{r}'$, keeping only the first term. The integral over the small sphere about \mathbf{r}' can be carried out analytically, by changing to a coordinate system centered at \mathbf{r}' ;

$$\mathbf{u} = \mathbf{r} - \mathbf{r}', \quad (8)$$

so that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) \approx f(\mathbf{r}') \lim_{a \rightarrow 0} \int_{u < R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}}. \quad (9)$$

We note that

$$\int_{u < R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}} = 4\pi \int_0^R du \frac{-3a^2 u^2}{(u^2 + a^2)^{5/2}} = 4\pi \frac{-R^3}{(R^2 + a^2)^{3/2}}. \quad (10)$$

If the infinitesimal value a is $a \ll R$, then $(R^2 + a^2)^{3/2} \approx R^3$ and the right hand side of Eq. 10 is -4π . Therefore, Eq. 9 becomes,

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) \approx f(\mathbf{r}')(-4\pi), \quad (11)$$

which is consistent with Eq. 5.