Notes for Lecture #15

Vector potentials in magnetostatics

The vector potential corresponding to a current density distribution $\mathbf{J}(\mathbf{r})$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1}$$

This expression is useful if the current density $\mathbf{J}(\mathbf{r})$ is confined within a finite region of space. Consider the following example corresponding to a rotating charged sphere of radius a, with ρ_0 denoting the uniform charge density within the sphere and ω denoting the angular rotation of the sphere:

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \rho_0\omega \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}'). \tag{3}$$

Noting that

$$\mathbf{r}' = r'\sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}}')\frac{\hat{\mathbf{x}}+\mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}')\frac{-\hat{\mathbf{x}}+\mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}}\right), \tag{4}$$

we see that the angular integral result takes the simple form:

$$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')\,\mathbf{r}' = \frac{r'}{r}\mathbf{r}\,\delta_{l1}. \tag{5}$$

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\rho_0\omega\times\mathbf{r}}{3r}\int_0^a dr'\, r'^3\frac{r_<}{r_>^2}, \tag{6}$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0\rho_0\omega\times\mathbf{r}}{3}\left(\frac{a^2}{2}-\frac{3r^2}{10}\right) & \text{for } r \leq a \\ \\ \frac{\mu_0\rho_0\omega\times\mathbf{r}}{3r^3}\frac{a^5}{5} & \text{for } r \geq a \end{cases}. \tag{7}$$

As another example, consider the current associated with an electron in a spherical atom. In this case, we assume that the current density is due to an electron in a bound atomic state with quantum numbers $|nlm_l\rangle$, as described by a wavefunction $\psi_{nlm_l}(\mathbf{r})$, where the azimuthal quantum number m_l is associated with a factor of the form $\mathrm{e}^{im_l\phi}$. For such a wavefunction the quantum mechanical current density operator can be evaluated:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i}\left(\psi^*_{nlm_l}\nabla\psi_{nlm_l} - \psi_{nlm_l}\nabla\psi^*_{nlm_l}\right). \tag{8}$$

Since the only complex part of this wavefunction is associated with the azimuthal quantum number, this can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e ir\sin\theta}\left(\psi^*_{nlm_l}\frac{\partial}{\partial\phi}\psi_{nlm_l} - \psi_{nlm_l}\frac{\partial}{\partial\phi}\psi^*_{nlm_l}\right)\hat{\phi} = \frac{-e\hbar m_l\hat{\phi}}{m_e r\sin\theta}\left|\psi_{nlm_l}\right|^2. \tag{9}$$

where m_e denotes the electron mass and e denotes the magnitude of the electron charge.

For example, consider the $|nlm = 211\rangle$ state of a H atom:

$$\psi_{211}(\mathbf{r}) = -\sqrt{\frac{1}{64\pi a^3}}\frac{r}{a}\mathrm{e}^{-r/(2a)}\sin\theta\mathrm{e}^{i\phi}, \tag{10}$$

and

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{64m_e\pi a^5}\mathrm{e}^{-r'/a}\,\hat{\mathbf{z}}\times\mathbf{r}', \tag{11}$$

where a here denotes the Bohr radius. Using arguments similar to those above, we find that

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}}\times\mathbf{r}}{192m_e\pi a^5 r}\int_0^\infty dr'\,r'^3\,\mathrm{e}^{-r'/a}\,\frac{r_<}{r_>^2}. \tag{12}$$

This expression can be integrated to give:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}}\times\mathbf{r}}{8m_e\pi r^3}\left[1 - \mathrm{e}^{-r/a}\left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3}\right)\right]. \tag{13}$$