January 16, 2005

Notes for Lecture #3

Form of Green's function solutions to the Poisson equation

According to Eq. 1.35 of your text for any two three-dimensional functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$\int_{\text{Vol}} \Big(\phi(\mathbf{r})\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2\phi(\mathbf{r})\Big)\, d^3r = \oint_{\text{Surf}} \left(\phi(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\phi(\mathbf{r})\right)\cdot \hat{\mathbf{r}} d^2r, \tag{1}$$

where $\hat{\mathbf{r}}$ denotes a unit vector normal to the integration surface. We can choose to evaluate this expression with $\phi(\mathbf{r}) = \Phi(\mathbf{r})$ (the electrostatic potential) and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$, and also make use of the identities:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{2}$$

and

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Then, the Green's identity (1) becomes

$$-4\pi \int_{\text{Vol}} \left(\Phi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r})}{4\pi\varepsilon_0}\right) d^3r = \oint_{\text{Surf}} \{\Phi(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla\Phi(\mathbf{r})\} \cdot \hat{\mathbf{r}} d^2r. \tag{4}$$

This expression can be further evaluated. If the arbitrary position, $\mathbf{r}'$ is included in the integration volume, then the equation (4) becomes

$$\Phi(\mathbf{r}') = \int_{\text{Vol}} G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r})}{4\pi\varepsilon_0} d^3r + \frac{1}{4\pi}\oint_{\text{Surf}} \{G(\mathbf{r}, \mathbf{r}')\nabla\Phi(\mathbf{r}) - \Phi(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}')\} \cdot \hat{\mathbf{r}} d^2r. \tag{5}$$

This expression is the same as Eq. 1.42 of your text if we switch the variables $\mathbf{r}' \Leftrightarrow \mathbf{r}$ and also use the fact that Green's function is symmetric in its arguments: $G(\mathbf{r}, \mathbf{r}') \equiv G(\mathbf{r}', \mathbf{r})$.

Mean value theorem for solutions to the Laplace equation

Consider an electrostatic field $\Phi(\mathbf{r})$ in a charge-free region so that it satisfies the Laplace equation:

$$\nabla^2\Phi(\mathbf{r}) = 0. \tag{6}$$

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots. \quad (7)$$

According to the premise of the theorem, we want to integrate both sides of the equation 7 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \quad (8)$$

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \quad (9)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u}\cdot\nabla = 0, \quad (10)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^2 = \frac{4\pi R^4}{3}\nabla^2, \quad (11)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^3 = 0, \quad (12)$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^4 = \frac{4\pi R^6}{5}\nabla^4. \quad (13)$$

Since $\nabla^2\Phi(\mathbf{r}) = 0$, the only non-zero term of the average it thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r}+\mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}), \quad (14)$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \; R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r}+\mathbf{u}). \quad (15)$$

Since this result is independent of the radius R, we see that we have proven the theorem.