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Notes for Lecture #36

Synchrotron Radiation

For this analysis we will use the geometry shown in Fig. 14.9 of **Jackson**. A particle with charge q is moving in a circular trajectory with radius ρ and speed v. Its trajectory as a function of time t is given by

$$\mathbf{r}_q(t) = \rho \sin(vt/\rho)\hat{\mathbf{x}} + \rho\left(1 - \cos(vt/\rho)\right)\hat{\mathbf{y}}. \tag{1}$$

Its velocity as a function time is given by

$$\mathbf{v}_q(t) = v\cos(vt/\rho)\hat{\mathbf{x}} + v\sin(vt/\rho)\hat{\mathbf{y}}. \tag{2}$$

The spectral intensity that we must evaluate is given by the expression (Eq. 14.67 in Jackson)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta)\mathrm{e}^{i\omega(t - \hat{\mathbf{r}}\cdot\mathbf{r}_q(t)/c)}dt\right|^2. \tag{3}$$

After some algebra, this expression can be put into the form

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_\parallel(\omega)|^2 + |C_\perp(\omega)|^2\right\}, \tag{4}$$

where the amplitude for the light polarized along the y-axis is given by

$$C_\parallel(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho)\mathrm{e}^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))} \tag{5}$$

and the amplitude for the light polarized perpendicular to $\hat{\mathbf{y}}$ and $\hat{\mathbf{r}}$ is given by

$$C_\perp(\omega) \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho)\mathrm{e}^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))}. \tag{6}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1 - 1/(2\gamma^2))$ passing a beam line port. In addition $\theta \approx 0$ and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency

$$\omega_c \equiv \frac{3c\gamma^3}{2\rho}. \tag{7}$$

The resultant intensity is then given by

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2\left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right\}. \tag{8}$$

By plotting this expression as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$.

The second example of synchroton radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of v/ρ. For this case we need to reconsider Eqs. 5 and 6. There is a very convenient Bessel function identity of the form:

$$\mathrm{e}^{-ia\sin\alpha} = \sum_{m=-\infty}^{\infty} J_m(a)\mathrm{e}^{-im\alpha}. \tag{9}$$

Here $J_m(a)$ is a Bessel function of integer order m. In our case $a = \frac{\omega\rho}{c}\cos\theta$ and $\alpha = \frac{vt}{\rho}$. Analyzing the "parallel" component we have

$$C_{\parallel} = \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty} dt\mathrm{e}^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}). \tag{10}$$

In determining this result, we have used the identity

$$\int_{-\infty}^{\infty} dt\mathrm{e}^{i(\omega-m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}). \tag{11}$$

Eq. 10 can be simplified to show that

$$C_{\parallel} = 2\pi i\sum_{-\infty}^{\infty} J'_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}), \tag{12}$$

where $J'_m(a) \equiv \frac{dJ_m(a)}{da}$. The "perpendicular" component can be analyzed in a similar way, using integration by parts to eliminate the extra $\cos(vt/\rho)$ term in the argument. The result is

$$C_{\perp} = 2\pi\frac{\tan\theta}{v/c}\sum_{-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}). \tag{13}$$

In both of these expressions, the sum over m includes both negative and positive values of m. However, only the positive values of ω and therefore positive values of m are of interest, and if we needed to use the negative m values, we could use the identity

$$J_{-m}(a) = (-1)^m J_m(a). \tag{14}$$

Combining these results, we find that the intensity spectrum for this case consists of a series of discrete frequencies which are multiples of v/ρ.

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{c}\sum_{m=0}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J'_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}. \tag{15}$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. It should have some implications for Astronomical observations, but I have not yet found any references for that.