Notes for Lecture #1

1 Introduction

1. Textbook and course structure
2. Motivation
3. Chapters I and 1 and Appendix of Jackson
   (a) Units - SI vs Gaussian
   (b) Laplace and Poisson Equations
   (c) Green’s Theorm

2 Units - SI vs Gaussian

Coulomb’s law has the form:
\[ F = K_C \frac{q_1 q_2}{r_{12}^2}. \]  \( (1) \)

Ampere’s law has the form:
\[ F = K_A \frac{i_1 i_2}{r_{12}^2} \frac{ds_1 \times ds_2 \times \hat{r}_{12}}{r_{12}^2}, \]  \( (2) \)

where the current and charge are related by \( i_1 = dq_1/dt \) for all unit systems. The two constants \( K_C \) and \( K_A \) are related so that their ratio \( K_C/K_A \) has the units of \((m/s)^2\) and it is experimentally known that in both the SI and CGS (Gaussian) unit systems, it has the value \( K_C/K_A = c^2 \), where \( c \) is the speed of light.

The choices for these constants in the SI and Gaussian units are given below:

<table>
<thead>
<tr>
<th></th>
<th>CGS (Gaussian)</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_C )</td>
<td>1</td>
<td>( \frac{1}{4\pi\epsilon_0} )</td>
</tr>
<tr>
<td>( K_A )</td>
<td>( \frac{1}{\mu_0 c^2} )</td>
<td>( \frac{\mu_0}{4\pi} )</td>
</tr>
</tbody>
</table>

Here, \( \frac{\mu_0}{4\pi} \equiv 10^{-7}N/A^2 \) and \( \frac{1}{4\pi\epsilon_0} = c^2 \cdot 10^{-7}N/A^2 = 8.98755 \times 10^9 N \cdot m^2/C^2 \).
Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

<table>
<thead>
<tr>
<th>Variable</th>
<th>SI</th>
<th>Gaussian</th>
<th>SI/Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unit</td>
<td>Relation</td>
<td>Unit</td>
</tr>
<tr>
<td>length</td>
<td>m</td>
<td>fundamental</td>
<td>cm</td>
</tr>
<tr>
<td>mass</td>
<td>kg</td>
<td>fundamental</td>
<td>gm</td>
</tr>
<tr>
<td>time</td>
<td>s</td>
<td>fundamental</td>
<td>s</td>
</tr>
<tr>
<td>force</td>
<td>N</td>
<td>kg \cdot m^2/s</td>
<td>dyne</td>
</tr>
<tr>
<td>current</td>
<td>A</td>
<td>fundamental</td>
<td>statampere</td>
</tr>
<tr>
<td>charge</td>
<td>C</td>
<td>A \cdot s</td>
<td>statcoulomb</td>
</tr>
</tbody>
</table>

One advantage of the Gaussian system is that all of the field vectors: \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{P}, \mathbf{M} \) have the same dimensions, and in vacuum, \( \mathbf{B} = \mathbf{H} \) and \( \mathbf{E} = \mathbf{D} \) and the dielectric and permittivity constants \( \epsilon \) and \( \mu \) are unitless.

<table>
<thead>
<tr>
<th>CGS (Gaussian)</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \cdot \mathbf{D} = 4\pi \rho )</td>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
</tr>
<tr>
<td>( \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} )</td>
<td>( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} )</td>
</tr>
<tr>
<td>( \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} )</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} )</td>
</tr>
<tr>
<td>( \mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) )</td>
<td>( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) )</td>
</tr>
<tr>
<td>( u = \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) )</td>
<td>( u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) )</td>
</tr>
<tr>
<td>( \mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}) )</td>
<td>( \mathbf{S} = (\mathbf{E} \times \mathbf{H}) )</td>
</tr>
</tbody>
</table>

“Proof” of the identity (Eq. (1.31))

\[
\nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}').
\] (3)
Noting that
\[ \int_{\text{small sphere about } r'} d^3r \, \delta^3(r - r') f(r) = f(r'), \quad (4) \]
we see that we must show that
\[ \int_{\text{small sphere about } r'} d^3r \, \nabla^2 \left( \frac{1}{|r - r'|} \right) f(r) = -4\pi f(r'). \quad (5) \]

We introduce a small radius \( a \) such that:
\[ \frac{1}{|r - r'|} = \lim_{a \to 0} \frac{1}{\sqrt{|r - r'|^2 + a^2}}. \quad (6) \]

For a fixed value of \( a \),
\[ \nabla^2 \frac{1}{\sqrt{|r - r'|^2 + a^2}} = \frac{-3a^2}{(|r - r'|^2 + a^2)^{5/2}}. \quad (7) \]

If the function \( f(r) \) is continuous, we can make a Taylor expansion of it about the point \( r = r' \), keeping only the first term. The integral over the small sphere about \( r' \) can be carried out analytically, by changing to a coordinate system centered at \( r' \):
\[ u = r - r', \quad (8) \]
so that
\[ \int_{\text{small sphere about } r'} d^3r \, \nabla^2 \left( \frac{1}{|r - r'|} \right) f(r) \approx f(r') \lim_{a \to 0} \int_{u < R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}}. \quad (9) \]

We note that
\[ \int_{u < R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}} = 4\pi \int_0^R du \frac{-3a^2u^2}{(u^2 + a^2)^{5/2}} = 4\pi \frac{-R^3}{(R^2 + a^2)^{3/2}}. \quad (10) \]

If the infinitesimal value \( a \) is \( a \ll R \), then \( (R^2 + a^2)^{3/2} \approx R^3 \) and the right hand side of Eq. 10 is \(-4\pi\). Therefore, Eq. 9 becomes,
\[ \int_{\text{small sphere about } r'} d^3r \, \nabla^2 \left( \frac{1}{|r - r'|} \right) f(r) \approx f(r')(-4\pi), \quad (11) \]
which is consistent with Eq. 5.