Notes for Lecture #11

Dipole fields

The dipole moment is defined by
\[ \mathbf{p} = \int d^3 r \rho(r) \mathbf{r}, \]
with the corresponding potential
\[ \Phi(r) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{r}}{r^2}, \]
and electrostatic field
\[ \mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{3\hat{r}(\mathbf{p} \cdot \hat{r}) - \mathbf{p}}{r^3} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right\}. \]

The last term of the field expression follows from the following derivation. We note that Eq. (3) is poorly defined as \( r \to 0 \), and consider the value of a small integral of \( \mathbf{E}(\mathbf{r}) \) about zero. (For this purpose, we are supposing that the dipole \( \mathbf{p} \) is located at \( \mathbf{r} = \mathbf{0} \).) In this case we will approximate
\[ \mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left( \int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3 \mathbf{r} \right) \delta^3(\mathbf{r}). \]

First we note that
\[ \int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 \mathbf{r} = -R^2 \int_{r = R} \Phi(\mathbf{r}) \hat{r} d\Omega. \]

This result follows from the Divergence theorem:
\[ \int_{\text{vol}} \nabla \cdot \mathbf{V} d^3 \mathbf{r} = \int_{\text{surface}} \mathbf{V} \cdot d\mathbf{A}. \]

In our case, this theorem can be used to prove Eq. (5) for each cartesian coordinate if we choose \( \mathbf{V} \equiv \hat{x} \Phi(\mathbf{r}) \) for the \( x \)- component for example:
\[ \int_{r \leq R} \nabla \Phi(\mathbf{r}) d^3 \mathbf{r} = \hat{x} \int_{r \leq R} \nabla \cdot (\hat{x} \Phi(\mathbf{r})) d^3 \mathbf{r} + \hat{y} \int_{r \leq R} \nabla \cdot (\hat{y} \Phi(\mathbf{r})) d^3 \mathbf{r} + \hat{z} \int_{r \leq R} \nabla \cdot (\hat{z} \Phi(\mathbf{r})) d^3 \mathbf{r}, \]
which is equal to
\[ \int_{r = R} \Phi(\mathbf{r}) R^2 d\Omega ((\hat{x} \cdot \hat{r}) \hat{x} + (\hat{y} \cdot \hat{r}) \hat{y} + (\hat{z} \cdot \hat{r}) \hat{z}) = \int_{r = R} \Phi(\mathbf{r}) R^2 d\Omega \hat{r}. \]

Thus,
\[ \int_{r \leq R} \mathbf{E}(\mathbf{r}) d^3 \mathbf{r} = -R^2 \int_{r = R} \Phi(\mathbf{r}) \hat{r} d\Omega. \]
Now, we notice that the electrostatic potential can be determined from the charge density \( \rho(\mathbf{r}) \) according to:

\[
\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' = \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi}{2l + 1} \int d^3\mathbf{r}' \rho(\mathbf{r}') \frac{r_<^{l+1}}{r_>^{l+1}} Y_{lm}^*(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{r}}).
\]

We also note that the unit vector can be written in terms of spherical harmonic functions:

\[
\hat{\mathbf{r}} = \left\{ \sin(\theta) \cos(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + \cos(\theta) \hat{z} \right\} \sqrt{\frac{4\pi}{3}} \left( Y_{1-1}(\hat{\mathbf{r}}) \frac{\hat{x} + i\hat{y}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}) \frac{-\hat{x} + i\hat{y}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}) \hat{z} \right).
\]

Therefore, when we evaluate the integral over solid angle \( \Omega \) in Eq. (5), only the \( l = 1 \) term contributes and the effect of the integration reduced to the expression:

\[
-R^2 \int_{r=\tilde{R}} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\varepsilon_0} \frac{4\pi R^2}{3} \int d^3\mathbf{r}' \rho(\mathbf{r}') \frac{r_<}{r_>^2} \hat{\mathbf{r}}'.
\]

The choice of \( r_< \) and \( r_> \) is a choice between the integration variable \( r' \) and the sphere radius \( R \). If the sphere encloses the charge distribution \( \rho(\mathbf{r}') \), then \( r_< = r' \) and \( r_> = R \) so that Eq. (12) becomes

\[
-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\varepsilon_0} \frac{4\pi R^2}{3} \frac{1}{R^2} \int d^3\mathbf{r}' \rho(\mathbf{r}') r'_\mathbf{r}' \equiv -\frac{\mathbf{P}}{3\varepsilon_0}.
\]

If the charge distribution \( \rho(\mathbf{r}') \) lies outside of the sphere, then \( r_> = r' \) and \( r_> = R \) so that Eq. (12) becomes

\[
-R^2 \int_{r=R} \Phi(\mathbf{r}) \hat{\mathbf{r}} d\Omega = -\frac{1}{4\pi\varepsilon_0} \frac{4\pi R^2}{3} R \int d^3\mathbf{r}' \rho(\mathbf{r}') r'\hat{\mathbf{r}}' \equiv \frac{4\pi R^3}{3} \mathbf{E}(0).
\]