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Notes for Lecture #19

Derivation of the Lienard-Wiechert potentials and fields

Consider a point charge q moving on a trajectory $R_q(t)$. We can write its charge density as

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \tag{1}$$

and the current density as

$$\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)), \tag{2}$$

where

$$\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}. \tag{3}$$

Evaluating the scalar and vector potentials in the Lorentz gauge,

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int\ d^3r'\ dt'\ \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\ \delta\left(t' - (t - |\mathbf{r}-\mathbf{r}'|/c)\right), \tag{4}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int\ d^3r'\ dt'\ \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\ \delta\left(t' - (t - |\mathbf{r}-\mathbf{r}'|/c)\right). \tag{5}$$

We performing the integrations over first d^3r' and then dt', and make use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\ f(t')\delta\left(t' - (t - |\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}}, \tag{6}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}. \tag{7}$$

We find

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}, \tag{8}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}, \tag{9}$$

where we have used the shorthand notation $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ and $\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$.

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \tag{10}$$

and

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t). \tag{11}$$

The trick of evaluating these derivatives is that the retarded time (7) depends on position $\mathbf{r}$ and on itself. We can show the following results using the shorthand notation defined above:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}, \tag{12}$$

and

$$\frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}. \tag{13}$$

Evaluating the gradient of the scalar potential, we find:

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1 - \frac{v^2}{c^2}\right) - \frac{\mathbf{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right], \tag{14}$$

and

$$-\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right]. \tag{15}$$

These results can be combined to determine the electric field:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{16}$$

We can also evaluate the curl of $\mathbf{A}$ to find the magnetic field:

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{17}$$

One can show that the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}. \tag{18}$$