Notes for Lecture #2

Examples of solutions of the one-dimensional Poisson equation

Consider the following one dimensional charge distribution:

\[
\rho(x) = \begin{cases} 
0 & \text{for } x < -a \\
-\rho_0 & \text{for } -a < x < 0 \\
+\rho_0 & \text{for } 0 < x < a \\
0 & \text{for } x > a 
\end{cases}
\]  

(1)

We want to find the electrostatic potential such that

\[
\frac{d^2\Phi(x)}{dx^2} = \frac{-\rho(x)}{\varepsilon_0},
\]

(2)

with the boundary condition \(\Phi(-\infty) = 0\).

The solution to the differential equation is given by:

\[
\Phi(x) = \begin{cases} 
0 & \text{for } x < -a \\
\frac{\rho_0}{2\varepsilon_0}(x + a)^2 & \text{for } -a < x < 0 \\
-\frac{\rho_0}{2\varepsilon_0}(x - a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\
\frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a 
\end{cases}
\]

(3)

The electrostatic field is given by:

\[
E(x) = \begin{cases} 
0 & \text{for } x < -a \\
-\frac{\rho_0}{\varepsilon_0}(x + a) & \text{for } -a < x < 0 \\
\frac{\rho_0}{\varepsilon_0}(x - a) & \text{for } 0 < x < a \\
0 & \text{for } x > a 
\end{cases}
\]

(4)

The electrostatic potential can be determined by piecewise solution within each of the four regions or by use of the Green’s function \(G(x, x') = 4\pi x_x\), where,

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} G(x, x')\rho(x')dx'.
\]

(5)

In the expression for \(G(x, x')\), \(x_x\) should be taken as the smaller of \(x\) and \(x'\). It can be shown that Eq. 5 gives the identical result for \(\Phi(x)\) as given in Eq. 3.
Notes on the one-dimensional Green’s functions

The Green’s function for the Poisson equation can be defined as a solution to the equation:

$$\nabla^2 G(x, x') = -4\pi \delta(x - x').$$  \hfill (6)

Here the factor of $4\pi$ is not really necessary, but ensures consistency with your text’s treatment of the 3-dimensional case. The meaning of this expression is that $x'$ is held fixed while taking the derivative with respect to $x$. It is easily shown that with this definition of the Green’s function (6), Eq. (5) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green’s function which satisfies Eq. (6), we notice that we can use two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0, \hfill (7)$$

where $i = 1$ or 2, to form

$$G(x, x') = \frac{4\pi}{W} \phi_1(x_\text{<}) \phi_2(x_\text{>}). \hfill (8)$$

This notation means that $x_\text{<}$ should be taken as the smaller of $x$ and $x'$ and $x_\text{>}$ should be taken as the larger. In this expression $W$ is the “Wronskian”:

$$W \equiv \frac{d\phi_1(x)}{dx} \phi_2(x) - \phi_1(x) \frac{d\phi_2(x)}{dx}. \hfill (9)$$

We can check that this “recipe” works by noting that for $x \neq x'$, Eq. (8) satisfies the defining equation 6 by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 7. The defining equation is singular at $x = x'$, but integrating 6 over $x$ in the neighborhood of $x'$ ($x' - \epsilon < x < x' + \epsilon$), gives the result:

$$\left[ \frac{dG(x, x')}{dx} \right]_{x=x'+\epsilon} - \left[ \frac{dG(x, x')}{dx} \right]_{x=x'-\epsilon} = -4\pi. \hfill (10)$$

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green’s function is as given above. For this piecewise continuous form of the Green’s function, the integration 5 can be evaluated:

$$\Phi(x) = \frac{1}{4\pi \varepsilon_0} \left\{ \int_{-\infty}^{x} G(x, x')\rho(x')dx' + \int_{x}^{\infty} G(x, x')\rho(x')dx' \right\}, \hfill (11)$$

which becomes

$$\Phi(x) = \frac{1}{\varepsilon_0} \left\{ \int_{-\infty}^{x} x'\rho(x')dx' + x \int_{x}^{\infty} \rho(x')dx' \right\}. \hfill (12)$$

Evaluating this expression, we find that we obtain the same result as given in Eq. (3).

In general, the Green’s function $G(x, x')$ solution (5) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all neutral charge densities, that is $\int_{-\infty}^{\infty} \rho(x)dx = 0$. 