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Notes for Lecture #2

Examples of solutions of the one-dimensional Poisson equation

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{1}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}, \tag{2}$$

with the boundary condition $\Phi(-\infty) = 0$.

The solution to the differential equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases}. \tag{3}$$

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}. \tag{4}$$

The electrostatic potential can be determined by piecewise solution within each of the four regions or by use of the Green's function $G(x, x') = 4\pi x_<$, where,

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x, x')\rho(x')dx'. \tag{5}$$

In the expression for $G(x, x')$, $x_<$ should be taken as the smaller of x and x'. It can be shown that Eq. 5 gives the identical result for $\Phi(x)$ as given in Eq. 3.

Notes on the one-dimensional Green's functions

The Green's function for the Poisson equation can be defined as a solution to the equation:

$$\nabla^2 G(x, x') = -4\pi\delta(x - x'). \tag{6}$$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x. It is easily shown that with this definition of the Green's function (6), Eq. (5) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green's function which satisfies Eq. (6), we notice that we can use two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0, \tag{7}$$

where $i = 1$ or 2, to form

$$G(x, x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>). \tag{8}$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger. In this expression W is the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{\phi_2(x)}{dx}. \tag{9}$$

We can check that this "recipe" works by noting that for $x \neq x'$, Eq. (8) satisfies the defining equation 6 by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 7. The defining equation is singular at $x = x'$, but integrating 6 over x in the neighborhood of x' $(x' - \epsilon < x < x' + \epsilon)$, gives the result:

$$\frac{dG(x, x')}{dx}\rfloor_{x=x'+\epsilon} - \frac{dG(x, x')}{dx}\rfloor_{x=x'-\epsilon} = -4\pi. \tag{10}$$

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green's function is as given above. For this piecewise continuous form of the Green's function, the integration 5 can be evaluated:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x, x')\rho(x')dx' + \int_{x}^{\infty} G(x, x')\rho(x')dx'\right\}, \tag{11}$$

which becomes

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}. \tag{12}$$

Evaluating this expression, we find that we obtain the same result as given in Eq. (3).

In general, the Green's function $G(x, x')$ solution (5) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all *neutral* charge densities, that is $\int_{-\infty}^{\infty} \rho(x)dx = 0$.