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# Notes for Lecture #24

**Electromagnetic wave guides**

In order to understand the operation of a wave guide, we must first learn how electromagnetic waves behave in a dissipative medium. A plane wave solution to Maxwell's equations of the form:

$$\mathbf{E} = \mathbf{E_0} e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t} \quad \text{and} \quad \mathbf{B} = \frac{k}{\omega}\hat{\mathbf{k}} \times \mathbf{E_0} e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t} \tag{1}$$

for the electric and magnetic fields, with the wave vector $k$ satisfying the relation:

$$k^2 = \omega^2 \mu\varepsilon \equiv \mathcal{R} + i\mathcal{I}. \tag{2}$$

We can determine the complex wavevector $k_r + ik_i$ according to

$$k_r = \left(\frac{\sqrt{\mathcal{R}^2 + \mathcal{I}^2} + \mathcal{R}}{2}\right)^{1/2} \quad \text{and} \quad k_i = \left(\frac{\sqrt{\mathcal{R}^2 + \mathcal{I}^2} - \mathcal{R}}{2}\right)^{1/2} \tag{3}$$

The form of the frequency dependent constants $\mathcal{R}$ and $\mathcal{I}$ depend on the materials. For the Drude model at low frequency (Eq. 7.56), $\mathcal{R} = \omega^2\mu\varepsilon_b$ and $\mathcal{I} = \omega\mu\sigma$, for example. The value of $k_i$ determines the rate of decay of the field amplitudes in the vicinity of the surface, with the skin depth given by $\delta \equiv 1/k_i$. In the limit that $\mathcal{I} \gg \mathcal{R}$, as in the case of a good conductor at low frequency, $\delta \approx (2/(\omega\mu\sigma))^{1/2}$.

For an "ideal" conductor $\mathcal{I} \to \infty$, so that the fields are confined to the surface. Because of the field continuity conditions at the surface of the conductor, this means that, $\mathbf{B}_{\text{tangential}} \neq 0$ (because there can be a surface current), $\mathbf{E}_{\text{normal}} \neq 0$ (because there can be a surface charge), but $\mathbf{B}_{\text{normal}} = 0$ and $\mathbf{E}_{\text{tangential}} = 0$.

Suppose we construct a wave guide from an "ideal" conductor, designating $\hat{\mathbf{z}}$ as the propagation direction. We will assume that the fields take the form:

$$\mathbf{E} = \mathbf{E}(x, y)e^{ikz-i\omega t} \quad \text{and} \quad \mathbf{B} = \mathbf{B}(x, y)e^{ikz-i\omega t} \tag{4}$$

inside the pipe, where now $k$ and $\varepsilon$ are assumed to be real. Assuming that there are no sources inside the pipe, the fields there must satisfy Maxwell's equations (8.16) which expand to the following :

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0. \tag{5}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0. \tag{6}$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \tag{7}$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y. \tag{8}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z. \tag{9}$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -i\mu\varepsilon\omega E_x. \tag{10}$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\varepsilon\omega E_y. \tag{11}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\varepsilon\omega E_z. \tag{12}$$

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\varepsilon\omega^2\right) E_x(x,y) = 0, \tag{13}$$

with similar expressions for each of the other field components. For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \tag{14}$$

with $k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$. From this result and Maxwell's equations, we can determine the other field components. For example Eqs. (7-8) simplify to

$$B_x = -\frac{k}{\omega} E_y \quad \text{and} \quad B_y = \frac{k}{\omega} E_x. \tag{15}$$

These results can be used in Eqs. (10-11) to solve for the fields $E_x$ and $E_y$ and $B_x$ and $B_y$:

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\varepsilon\omega^2} \frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{16}$$

and

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\varepsilon\omega^2} \frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right). \tag{17}$$

One can check this result to show that these results satisfy the boundary conditions. For example, $\mathbf{E}_{\text{tangential}} = 0$ is satisfied since $E_x(x,0) = E_x(x,b) = 0$ and $E_y(0,y) = E_y(a,y) = 0$. This was made possible choosing $\nabla B_z\rfloor_{\text{surface}} \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ denotes a unit normal vector pointing out of the wave guide surface.