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Notes for Lecture #27

Two formulations of electromagnetic fields produced by a charged particle moving at constant velocity

In Chapter 11 of **Jackson** (page 559 – Eqs. 11.151-2 and Fig. 11.8), we derived the electric and magnetic field of a particle having charge q moving at velocity v along the $\hat{\mathbf{x}}_{\mathbf{1}}$ axis. The results are for the fields at the point $\mathbf{r} = b\hat{\mathbf{x}}_{\mathbf{2}}$ are:

$$\mathbf{E}(x_1, x_2, x_3, t) = \mathbf{E}(0, b, 0, t) = q\frac{-v\gamma t\hat{\mathbf{x}}_{\mathbf{1}} + \gamma b\hat{\mathbf{x}}_{\mathbf{2}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}} \tag{1}$$

and

$$\mathbf{B}(x_1, x_2, x_3, t) = \mathbf{B}(0, b, 0, t) = q\frac{\gamma\beta b\hat{\mathbf{x}}_{\mathbf{3}}}{\left(b^2 + (v\gamma t)^2\right)^{3/2}} \tag{2}$$

for the electric and magnetic fields respectively. The denominators of these expressions are easily interpreted as the distance of the particle from the field point, as measured in the particle's own reference frame. On the other hand, in Chapter 6 we considered the same physical problem from the point of view of Liénard-Wiechert potentials:

Consider the electric field produced by a point charge q moving on a trajectory described by $\mathbf{r_0(t)}$ with $\rho(r,t) \equiv q\delta^3(\mathbf{r} - \mathbf{r_0}(t))$. Assume that $\mathbf{v_0}(t) \equiv \partial\mathbf{r_0}(t)/\partial t$ and $\partial^2\mathbf{r_0}(t)/\partial t^2 = 0$. Show that the electric field can be written in the form:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{(1 - v_0^2/c^2)(\mathbf{R} - \mathbf{v_0}R/c)}{(R - \mathbf{v_0}\cdot\mathbf{R}/c)^3} \underset{\text{Gaussian units}}{\longrightarrow} q\frac{(1 - v_0^2/c^2)(\mathbf{R} - \mathbf{v_0}R/c)}{(R - \mathbf{v_0}\cdot\mathbf{R}/c)^3}, \tag{3}$$

where $R \equiv |\mathbf{R}(t_r)|$, $\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{r_0}(t_r)$, and where all quantities which depend on time on the right hand side of the equation are evaluated at the retarded time $t_r \equiv t - R(t_r)/c$.

In the same notation, the magnetic field is given by

$$\mathbf{B} = \frac{\mathbf{R} \times \mathbf{E}}{R}. \tag{4}$$

If we evaluate this result for the same case as above (Fig. 11.8 of **Jackson**), $\mathbf{v_0} \equiv v\hat{\mathbf{x}}_{\mathbf{1}}$, and $\mathbf{R}(t_r) = -vt_r\hat{\mathbf{x}}_{\mathbf{1}} + b\hat{\mathbf{x}}_{\mathbf{2}}$. In order to relate this result to Eqs. 1 and 2 above, we need to express t_r in terms of the known quantities. Noting that

$$R(t_r) = c(t - t_r) = \sqrt{(vt_r)^2 + b^2}, \tag{5}$$

we find that t_r must be a solution to the quadratic equation:

$$t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2/c^2 = 0 \tag{6}$$

with the physical solution:

$$t_r = \gamma \left(\gamma t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c} \right). \tag{7}$$

Now we can express the length parameter which appears in Eq. 3 as

$$R = \gamma \left(-\beta v \gamma t + \sqrt{(v\gamma t)^2 + b^2} \right). \tag{8}$$

We also can show that the numerator of Eq. 3 can be evaluated:

$$\mathbf{R} - \mathbf{v_0} R/c = -vt\hat{\mathbf{x}}_\mathbf{1} + b\hat{\mathbf{x}}_\mathbf{2}, \tag{9}$$

and the denominator can be evaluated:

$$R - \mathbf{v_0} \cdot \mathbf{R}/c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}. \tag{10}$$

Substituting these results into Eqs. 3 and 4, we obtain the same electric and magnetic fields as given in Eqs. 1 and 2 from the field transformation approach.