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Notes for Lecture #6

The Green's function allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S [G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')]\cdot\hat{\mathbf{r}}'d^2r'. \quad (1)$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. For some special cases, we can use the results of the method of images to construct Dirichlet Green's functions as described in Section 2.6 of your text.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x)u_m(x)\,dx = \delta_{nm}. \quad (2)$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x)u_n(x') = \delta(x-x'). \quad (3)$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2}G(x,x') = 4\pi\delta(x-x'). \quad (4)$$

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x), \quad (5)$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write the Green's functions as

$$G(x,x') = 4\pi\sum_n \frac{u_n(x)u_n(x')}{\alpha_n}. \quad (6)$$

For example, if $u_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, then

$$G(x,x') = \frac{8\pi}{a}\sum_n \frac{\sin(n\pi x/a)\sin(n\pi x'/a)}{\left(\frac{n\pi}{a}\right)^2}. \quad (7)$$

These ideas can easily be extended to two and three dimensions. For example if $\{u_n(x)\}$, $\{v_n(x)\}$, and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written:

$$G(x, x', y, y', z, z') = 4\pi \sum_{lmn} \frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n}, \tag{8}$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y), \quad \text{and} \quad \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z). \tag{9}$$

See Eq. 3.167 in **Jackson** for an example.

An alternative method of finding Green's functions for second order ordinary differential equations is based on a product of two independent solutions of the homogeneous equation, $u_1(x)$ and $u_2(x)$, which satisfy the boundary conditions at x_1 and x_1, respectively:

$$G(x, x') = K u_1(x_<)u_2(x_>), \quad \text{where} \quad K \equiv \frac{4\pi}{u_1 \frac{du_2}{dx} - \frac{du_1}{dx}u_2}, \tag{10}$$

with $x_<$ meaning the smaller of x and x' and $x_>$ meaning the larger of x and x'. For example, we have previously discussed the example of the one dimensional Poisson equation with the boundary condition $\Phi(0) = 0$ and $\frac{d\Phi(\infty)}{dx} = 0$ to have the form:

$$G(x, x') = -4\pi x_<. \tag{11}$$

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of **Jackson**. For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x, x', y, y') = \sum_n u_n(x)u_n(x')g_n(y, y'). \tag{12}$$

If the functions $\{u_n(x)\}$ satisfy Eq. 5, then we must require that G satisfy the equation:

$$\nabla^2 G = \sum_n u_n(x)u_n(x') \left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(x - x')\delta(y - y'). \tag{13}$$

The $y-$dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'), \tag{14}$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0, \tag{15}$$

and a constant related to the Wronskian:

$$K_n \equiv \frac{4\pi}{v_{n_1}\frac{dv_{n_2}}{dy} - \frac{dv_{n_1}}{dy}v_{n_2}}. \tag{16}$$

If these functions also satisfy the appropriate boundary conditions, we can then construct the 2-dimensional Green's function from

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') K_n v_{n_1}(y_<) v_{n_2}(y_>). \tag{17}$$

For example, a Green's function for a two-dimensional rectangular system with $0 \le x \le a$ and $0 \le y \le b$, which vanishes on each of the boundaries can be expanded:

$$G(x, x', y, y') = 8 \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sinh\left(\frac{n\pi y_<}{a}\right) \sinh\left(\frac{n\pi}{a}(b - y_>)\right)}{n \sinh\left(\frac{n\pi b}{a}\right)}. \tag{18}$$

As an example, we can use this result to solve the 2-dimensional Laplace equation in the square region $0 \le x \le 1$ and $0 \le y \le 1$ with the boundary condition $\Phi(x, 0) = \Phi(0, y) = \Phi(1, y) = 0$ and $\Phi(x, 1) = V_0$. In this case, in determining $\Phi(x, y)$ using Eq. (1) there is no volume contribution (since the charge is zero) and the "surface" integral becomes a line integral $0 \le x' \le 1$ for $y' = 1$. Using the form from Eq. (18) with $a = b = 1$, it can be shown that the result takes the form:

$$\Phi(x, y) = \sum_{n=0}^{\infty} 4V_0 \quad \frac{\sin[(2n+1)\pi x] \sinh[(2n+1)\pi y]}{(2n+1)\pi \sinh[(2n+1)\pi]} \tag{19}$$