Notes for Lecture #35

Electromagnetic wave guides

In order to understand the operation of a wave guide, we must first learn how electromagnetic waves behave in a dissipative medium. A plane wave solution to Maxwell’s equations of the form:

\[ E = E_0 e^{i(kr - i\omega t)} \quad \text{and} \quad B = \frac{k}{\omega} \hat{\kappa} \times E_0 e^{i(kr - i\omega t)} \quad (1) \]

for the electric and magnetic fields, with the wave vector \( k \) satisfying the relation:

\[ k^2 = \omega^2 \mu \varepsilon \equiv R + iI. \quad (2) \]

We can determine the complex wavevector \( k_r + ik_i \) according to

\[ k_r = \left( \frac{\sqrt{R^2 + I^2} + R}{2} \right)^{1/2} \quad \text{and} \quad k_i = \left( \frac{\sqrt{R^2 + I^2} - R}{2} \right)^{1/2} \quad (3) \]

The form of the frequency dependent constants \( R \) and \( I \) depend on the materials. For the Drude model at low frequency (Eq. 7.56), \( R = \omega^2 \mu \varepsilon_b \) and \( I = \omega \mu \sigma \), for example. The value of \( k_i \) determines the rate of decay of the field amplitudes in the vicinity of the surface, with the skin depth given by \( \delta \equiv 1/k_i \). In the limit that \( I \gg R \), as in the case of a good conductor at low frequency, \( \delta \approx (2/(\omega \mu \sigma))^{1/2} \).

For an "ideal" conductor \( I \to \infty \), so that the fields are confined to the surface. Because of the field continuity conditions at the surface of the conductor, this means that, \( \mathbf{B}_\text{tangential} \neq 0 \) (because there can be a surface current), \( \mathbf{E}_\text{normal} \neq 0 \) (because there can be a surface charge), but \( \mathbf{B}_\text{normal} = 0 \) and \( \mathbf{E}_\text{tangential} = 0 \).

Suppose we construct a wave guide from an "ideal" conductor, designating \( \hat{z} \) as the propagation direction. We will assume that the fields take the form:

\[ E = E(x, y)e^{ikz - i\omega t} \quad \text{and} \quad B = B(x, y)e^{ikz - i\omega t} \quad (4) \]

inside the pipe, where now \( k \) and \( \varepsilon \) are assumed to be real. Assuming that there are no sources inside the pipe, the fields there must satisfy Maxwell’s equations (8.16) which expand to the following:

\[ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0. \quad \text{(5)} \]

\[ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0. \quad \text{(6)} \]

\[ \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \quad \text{(7)} \]
\[ i k E_x - \frac{\partial E_z}{\partial x} = i \omega B_y. \]  \hspace{1cm} (8)

\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i \omega B_z. \]  \hspace{1cm} (9)

\[ \frac{\partial B_z}{\partial y} - i k B_y = -i \mu \varepsilon \omega E_x. \]  \hspace{1cm} (10)

\[ i k B_x - \frac{\partial B_z}{\partial x} = -i \mu \varepsilon \omega E_y. \]  \hspace{1cm} (11)

\[ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i \mu \varepsilon \omega E_z. \]  \hspace{1cm} (12)

Combining Faraday’s Law and Ampere’s Law, we find that each field component must satisfy a two-dimensional Helmholtz equation:

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu \varepsilon \omega^2 \right) E_x(x, y) = 0, \]  \hspace{1cm} (13)

with similar expressions for each of the other field components. For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

\[ E_z(x, y) \equiv 0 \text{ and } B_z(x, y) = B_0 \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right), \]  \hspace{1cm} (14)

with \( k^2 \equiv k_{mn}^2 = \mu \varepsilon \omega^2 - \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right] \). From this result and Maxwell’s equations, we can determine the other field components. For example Eqs. (7-8) simplify to

\[ B_x = -\frac{k}{\omega} E_y \text{ and } B_y = \frac{k}{\omega} E_x. \]  \hspace{1cm} (15)

These results can be used in Eqs. (10-11) to solve for the fields \( E_x \) and \( E_y \) and \( B_x \) and \( B_y \):

\[ E_x = \frac{\omega}{k} B_y = \frac{-i \omega}{k^2 - \mu \varepsilon \omega^2} \frac{\partial B_z}{\partial y} = \frac{-i \omega}{k^2 - \mu \varepsilon \omega^2} \frac{n \pi}{b} B_0 \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right), \]  \hspace{1cm} (16)

and

\[ E_y = -\frac{\omega}{k} B_x = \frac{i \omega}{k^2 - \mu \varepsilon \omega^2} \frac{\partial B_z}{\partial x} = \frac{i \omega}{k^2 - \mu \varepsilon \omega^2} \frac{m \pi}{a} B_0 \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right). \]  \hspace{1cm} (17)

One can check this result to show that these results satisfy the boundary conditions. For example, \( \mathbf{E}_{\text{tangential}} = 0 \) is satisfied since \( E_x(x, 0) = E_x(x, b) = 0 \) and \( E_y(0, y) = E_y(a, y) = 0 \). This was made possible choosing \( \nabla B_z \mid_{\text{surface}} \mathbf{n} = 0 \), where \( \mathbf{n} \) denotes a unit normal vector pointing out of the wave guide surface.