Notes for Lecture #33

Synchrotron Radiation

For this analysis we will use the geometry shown in Fig. 14.9 of Jackson. A particle with charge $q$ is moving in a circular trajectory with radius $\rho$ and speed $v$. Its trajectory as a function of time $t$ is given by

$$R_q(t) = \rho \sin(vt/\rho) \hat{x} + \rho (1 - \cos(vt/\rho)) \hat{y}. \quad (1)$$

Its velocity as a function time is given by

$$v_q(t) = v \cos(vt/\rho) \hat{x} + v \sin(vt/\rho) \hat{y}. \quad (2)$$

The spectral intensity that we must evaluate is given by the expression (Eq. 14.67 in Jackson)

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2 \omega^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} \hat{r} \times (\hat{r} \times \beta)e^{i\omega(t-\hat{r} \cdot R_q(t)/c)} dt \right|^2. \quad (3)$$

After some algebra, this expression can be put into the form

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{4\pi^2c} \left\{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \right\}, \quad (4)$$

where the amplitude for the light polarized along the $y$-axis is given by

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho)e^{i\omega(t-\frac{\rho c}{\gamma} \cos \theta \sin(vt/\rho))} \quad (5)$$

and the amplitude for the light polarized perpendicular to $\hat{y}$ and $\hat{r}$ is given by

$$C_{\perp}(\omega) \int_{-\infty}^{\infty} dt \sin \theta \cos(vt/\rho)e^{i\omega(t-\frac{\rho c}{\gamma} \cos \theta \sin(vt/\rho))}. \quad (6)$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1 - 1/(2\gamma^2))$) passing a beam line port. In addition $\theta \approx 0$ and the relevant integration times $t$ are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency

$$\omega_{c} \equiv \frac{3c\gamma^3}{2\rho}. \quad (7)$$

The resultant intensity is then given by

$$\frac{d^2I}{d\omega d\Omega} = \frac{3q^2 \gamma^2 (\omega/\omega_c)^2 (1+\gamma^2 \theta^2)^2}{4\pi^2c} \left\{ \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1+\gamma^2 \theta^2)^{3/2} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1+\gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1+\gamma^2 \theta^2)^{3/2} \right) \right]^2 \right\}. \quad (8)$$
By plotting this expression as a function of $\omega$, we see that the intensity is largest near $\omega \approx \omega_c$.

The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of $v/\rho$. For this case we need to reconsider Eqs. 5 and 6. There is a very convenient Bessel function identity of the form:

$$e^{-ia\sin \alpha} = \sum_{m=-\infty}^{\infty} J_m(a)e^{-ima}.$$  \hspace{1cm} (9)

Here $J_m(a)$ is a Bessel function of integer order $m$. In our case $a = \frac{\omega \rho}{c} \cos \theta$ and $\alpha = \frac{vt}{\rho}$.

Analyzing the “parallel” component we have

$$C_\parallel = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))} = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \sum_{-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) 2\pi \delta(\omega - m\frac{v}{\rho}).$$ \hspace{1cm} (10)

In determining this result, we have used the identity

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho}) t} = 2\pi \delta(\omega - m\frac{v}{\rho}).$$ \hspace{1cm} (11)

Eq. 10 can be simplified to show that

$$C_\parallel = 2\pi i \sum_{-\infty}^{\infty} J'_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m\frac{v}{\rho}),$$ \hspace{1cm} (12)

where $J'_m(a) \equiv \frac{dJ_m(a)}{da}$. The “perpendicular” component can be analyzed in a similar way, using integration by parts to eliminate the extra $\cos(vt/\rho)$ term in the argument. The result is

$$C_\perp = 2\pi \tan \theta \frac{v}{c} \sum_{-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m\frac{v}{\rho}).$$ \hspace{1cm} (13)

In both of these expressions, the sum over $m$ includes both negative and positive values of $m$. However, only the positive values of $\omega$ and therefore positive values of $m$ are of interest, and if we needed to use the negative $m$ values, we could use the identity

$$J_{-m}(a) = (-1)^m J_m(a).$$ \hspace{1cm} (14)

Combining these results, we find that the intensity spectrum for this case consists of a series of discrete frequencies which are multiples of $v/\rho$.

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{c} \sum_{m=0}^{\infty} \delta(\omega - m\frac{v}{\rho}) \left\{ J'_m \left( \frac{\omega \rho}{c} \cos \theta \right)^2 + \frac{\tan^2 \theta}{v^2/c^2} J_m \left( \frac{\omega \rho}{c} \cos \theta \right)^2 \right\}.\hspace{1cm} (15)$$

These results were derived by Julian Schwinger (Phys. Rev. 75, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson’s text. It should have some implications for Astronomical observations, but I have not yet found any references for that.