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Notes for Lecture #5

Interesting properties of the Poisson and Laplace Equations

Mean value theorem for solutions to the Laplace equation

Consider an electrostatic field $\Phi(\mathbf{r})$ in a charge-free region so that it satisfies the Laplace equation:

$$\nabla^2\Phi(\mathbf{r}) = 0. \tag{1}$$

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots. \tag{2}$$

According to the premise of the theorem, we want to integrate both sides of the equation 2 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \tag{3}$$

We note that

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \tag{4}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\mathbf{u}\cdot\nabla = 0, \tag{5}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^2 = \frac{4\pi R^4}{3}\nabla^2, \tag{6}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^3 = 0, \tag{7}$$

and

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^4 = \frac{4\pi R^6}{5}\nabla^4. \tag{8}$$

Since $\nabla^2\Phi(\mathbf{r}) = 0$, the only non-zero term of the average it thus the first term:

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) = 4\pi R^2\Phi(\mathbf{r}), \tag{9}$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \;\; R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) \equiv \frac{1}{4\pi R^2} \int_{\text{sphere}} dS_u \Phi(\mathbf{r}+\mathbf{u}). \tag{10}$$

Since this result is independent of the radius R, we see that we have proven the theorem.

Form of Green's function solutions to the Poisson equation

According to Eq. 1.35 of your text for any two three-dimensional functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$\int_{\text{Vol}} \Big(\phi(\mathbf{r})\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2\phi(\mathbf{r})\Big)\, d^3r = \oint_{\text{Surf}} (\phi(\mathbf{r})\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla\phi(\mathbf{r})) \cdot \hat{\mathbf{r}} d^2r, \tag{11}$$

where $\hat{\mathbf{r}}$ denotes a unit vector normal to the integration surface. We can choose to evaluate this expression with $\phi(\mathbf{r}) = \Phi(\mathbf{r})$ (the electrostatic potential) and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}')$, and also make use of the identities:

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{12}$$

and

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{13}$$

Then, the Green's identity (11) becomes

$$-4\pi \int_{\text{Vol}} \left(\Phi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r})}{4\pi\varepsilon_0}\right) d^3r = \oint_{\text{Surf}} \{\Phi(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla\Phi(\mathbf{r})\} \cdot \hat{\mathbf{r}} d^2r. \tag{14}$$

This expression can be further evaluated. If the arbitrary position, $\mathbf{r}'$ is included in the integration volume, then the equation (14) becomes

$$\Phi(\mathbf{r}') = \int_{\text{Vol}} G(\mathbf{r}, \mathbf{r}')\frac{\rho(\mathbf{r})}{4\pi\varepsilon_0} d^3r + \frac{1}{4\pi} \oint_{\text{Surf}} \{G(\mathbf{r}, \mathbf{r}')\nabla\Phi(\mathbf{r}) - \Phi(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}')\} \cdot \hat{\mathbf{r}} d^2r. \tag{15}$$

This expression is the same as Eq. 1.42 of your text if we switch the variables $\mathbf{r}' \Leftrightarrow \mathbf{r}$ and also use the fact that Green's function is symmetric in its arguments: $G(\mathbf{r}, \mathbf{r}') \equiv G(\mathbf{r}', \mathbf{r})$.