

Notes for Lecture #15

Vector potentials in magnetostatics

The vector potential which vanishes at infinity and corresponds to a confined current density distribution $\mathbf{J}(\mathbf{r})$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1)$$

This expression is useful if the current density $\mathbf{J}(\mathbf{r})$ is confined within a finite region of space. Consider the following example corresponding to a rotating charged sphere of radius a , with ρ_0 denoting the uniform charge density within the sphere and ω denoting the angular rotation of the sphere:

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \rho_0 \omega \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}'). \quad (3)$$

Noting that

$$\mathbf{r}' = r' \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{-\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}') \hat{\mathbf{z}} \right), \quad (4)$$

we see that the angular integral result takes the simple form:

$$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}') \mathbf{r}' = \frac{r'}{r} \mathbf{r} \delta_{l1}. \quad (5)$$

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3r} \int_0^a dr' r'^3 \frac{r_{<}}{r_{>}^2}, \quad (6)$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3r^3} \frac{a^5}{5} & \text{for } r \geq a \end{cases}. \quad (7)$$

As another example, consider the current associated with an electron in a spherical atom. In this case, we assume that the current density is due to an electron in a bound atomic state with quantum numbers $|nlm_l\rangle$, as described by a wavefunction $\psi_{nlm_l}(\mathbf{r})$, where the azimuthal

quantum number m_l is associated with a factor of the form $e^{im_l\phi}$. For such a wavefunction the quantum mechanical current density operator can be evaluated:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i} \left(\psi_{nlm_l}^* \nabla \psi_{nlm_l} - \psi_{nlm_l} \nabla \psi_{nlm_l}^* \right). \quad (8)$$

Since the only complex part of this wavefunction is associated with the azimuthal quantum number, this can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i r \sin \theta} \left(\psi_{nlm_l}^* \frac{\partial}{\partial \phi} \psi_{nlm_l} - \psi_{nlm_l} \frac{\partial}{\partial \phi} \psi_{nlm_l}^* \right) \hat{\phi} = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin \theta} |\psi_{nlm_l}|^2. \quad (9)$$

where m_e denotes the electron mass and e denotes the magnitude of the electron charge.

For example, consider the $|nlm = 211\rangle$ state of a H atom:

$$\psi_{211}(\mathbf{r}) = -\sqrt{\frac{1}{64\pi a^3}} \frac{r}{a} e^{-r/(2a)} \sin \theta e^{i\phi}, \quad (10)$$

and

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{64m_e\pi a^5} e^{-r'/a} \hat{\mathbf{z}} \times \mathbf{r}', \quad (11)$$

where a here denotes the Bohr radius. Using arguments similar to those above, we find that

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{192m_e\pi a^5 r} \int_0^\infty dr' r'^3 e^{-r'/a} \frac{r_{<}}{r_{>}^2}. \quad (12)$$

This expression can be integrated to give:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{8m_e\pi r^3} \left[1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]. \quad (13)$$