## Notes for Lecture #29

## Synchrotron Radiation

For this analysis we will use the geometry shown in Fig. 14.9 of **Jackson**. A particle with charge q is moving in a circular trajectory with radius  $\rho$  and speed v. Its trajectory as a function of time t is given by

$$\mathbf{R}_{q}(t) = \rho \sin(vt/\rho)\hat{\mathbf{x}} + \rho \left(1 - \cos(vt/\rho)\right)\hat{\mathbf{y}}.\tag{1}$$

Its velocity as a function time is given by

$$\mathbf{v}_{q}(t) = v\cos(vt/\rho)\hat{\mathbf{x}} + v\sin(vt/\rho)\hat{\mathbf{y}}.$$
 (2)

The spectral intensity that we must evaluate is given by the expression (Eq. 14.67 in Jackson)

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \beta) e^{i\omega(t - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t)/c)} dt \right|^2.$$
 (3)

After some algebra, this expression can be put into the form

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2c} \left\{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \right\},\tag{4}$$

where the amplitude for the light polarized along the y-axis is given by

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$
 (5)

and the amplitude for the light polarized perpendicular to  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{r}}$  is given by

$$C_{\perp}(\omega) \int_{-\infty}^{\infty} dt \sin \theta \cos(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}.$$
 (6)

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light  $(v \approx c(1 - 1/(2\gamma^2)))$  passing a beam line port. In addition, because of the design of the radiation ports,  $\theta \approx 0$ , and the relevant integration times t are close to  $t \approx 0$ . This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency

$$\omega_c \equiv \frac{3c\gamma^3}{2\rho}.\tag{7}$$

The resultant intensity is then given by

$$\frac{d^{2}I}{d\omega d\Omega} = \frac{3q^{2}\gamma^{2}}{4\pi^{2}c} \left(\frac{\omega}{\omega_{c}}\right)^{2} (1+\gamma^{2}\theta^{2})^{2} \left\{ \left[K_{2/3} \left(\frac{\omega}{2\omega_{c}} (1+\gamma^{2}\theta^{2})^{\frac{3}{2}}\right)\right]^{2} + \frac{\gamma^{2}\theta^{2}}{1+\gamma^{2}\theta^{2}} \left[K_{1/3} \left(\frac{\omega}{2\omega_{c}} (1+\gamma^{2}\theta^{2})^{\frac{3}{2}}\right)\right]^{2} \right\}. \tag{8}$$

By plotting this expression as a function of  $\omega$ , we see that the intensity is largest near  $\omega \approx \omega_c$ .

The second example of synchroton radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of  $v/\rho$ . For this case we need to reconsider Eqs. 5 and 6. There is a very convenient Bessel function identity of the form:

$$e^{-ia\sin\alpha} = \sum_{m=-\infty}^{\infty} J_m(a)e^{-im\alpha}.$$
 (9)

Here  $J_m(a)$  is a Bessel function of integer order m. In our case  $a = \frac{\omega \rho}{c} \cos \theta$  and  $\alpha = \frac{vt}{\rho}$ . Analyzing the "parallel" component we have

$$C_{\parallel} = \frac{c}{-i\omega\rho} \frac{\partial}{\partial\cos\theta} \int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \frac{c}{-i\omega\rho} \frac{\partial}{\partial\cos\theta} \sum_{-\infty}^{\infty} J_m \left(\frac{\omega\rho}{c}\cos\theta\right) 2\pi\delta(\omega - m\frac{v}{\rho}). \tag{10}$$

In determining this result, we have used the identity

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}). \tag{11}$$

Eq. 10 can be simplified to show that

$$C_{\parallel} = 2\pi i \sum_{-\infty}^{\infty} J_m' \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m \frac{v}{\rho}), \tag{12}$$

where  $J'_m(a) \equiv \frac{dJ_m(a)}{da}$ . The "perpendicular" component can be analyzed in a similar way, using integration by parts to eliminate the extra  $\cos(vt/\rho)$  term in the argument. The result is

$$C_{\perp} = 2\pi \frac{\tan \theta}{v/c} \sum_{-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m \frac{v}{\rho}). \tag{13}$$

In both of these expressions, the sum over m includes both negative and positive values of m. However, only the positive values of  $\omega$  and therefore positive values of m are of interest, and if we needed to use the negative m values, we could use the identity

$$J_{-m}(a) = (-1)^m J_m(a). (14)$$

Combining these results, we find that the intensity spectrum for this case consists of a series of discrete frequencies which are multiples of  $v/\rho$ .

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{c} \sum_{m=0}^{\infty} \delta(\omega - m\frac{v}{\rho}) \left\{ \left[ J_m' \left( \frac{\omega\rho}{c} \cos\theta \right) \right]^2 + \frac{\tan^2\theta}{v^2/c^2} \left[ J_m \left( \frac{\omega\rho}{c} \cos\theta \right) \right]^2 \right\}. \tag{15}$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. It should have some implications for Astronomical observations, but I have not yet found any references for that. For information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron\_sources.html.