Notes on transformations of Spherical Harmonic functions

These notes use the convention of M. E. Rose, *Elementary Theory of Angular Momentum*, John Wiley & Sons, Inc. 1957 which seems to be consistent with your Tinkham text. Consider a transformation of the spherical harmonic functions:

$$Y_{lm}(\widehat{\mathcal{R}\mathbf{r}}) = \sum_{m'} Y_{lm'}(\hat{\mathbf{r}})\mathcal{D}^l_{m'm}(\mathcal{R}) \tag{1}$$

Here the transformation $\mathcal{R}\mathbf{r}$ might be a rotation through the 3 Euler angles (α about the $\hat{\mathbf{z}}$ axis, β about the new $\hat{\mathbf{y}}'$ axis, and γ about the new $\hat{\mathbf{z}}$" axis) so that

$$\mathcal{R}\mathbf{r} = M_{z"}(\gamma)M_{y'}(\beta)M_z(\alpha)\mathbf{r}. \tag{2}$$

$$M_{z"}(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

$$M_{y'}(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}, \tag{4}$$

and

$$M_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

By multiplying these three matrices, we find the 9 components of the rotation matrix to be:

$$\mathcal{R}_{xx} = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma \tag{6}$$

$$\mathcal{R}_{xy} = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma \tag{7}$$

$$\mathcal{R}_{xz} = -\sin\beta\cos\gamma \tag{8}$$

$$\mathcal{R}_{yx} = -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma \tag{9}$$

$$\mathcal{R}_{yy} = -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma \tag{10}$$

$$\mathcal{R}_{yz} = \sin\beta\sin\gamma \tag{11}$$

$$\mathcal{R}_{zx} = \cos\alpha\sin\beta \tag{12}$$

$$\mathcal{R}_{zy} = \sin\alpha\sin\beta \tag{13}$$

$$\mathcal{R}_{zz} = \cos\beta \tag{14}$$

It can be shown that the spherical harmonic transformation representation takes the form:

$$\mathcal{D}^l_{m'm}(\mathcal{R}) \equiv \mathcal{D}^l_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'} d^l_{m'm}(\cos\beta)e^{-i\gamma m}, \tag{15}$$

For $m' \geq m$,

$$d^l_{m'm}(\cos\beta) = \sqrt{\frac{(l-m)!(l+m')!}{(l+m)!(l-m')!}}\frac{1}{(m'-m)!}\left(\cos\frac{\beta}{2}\right)^{2l-(m'-m)}\left(\sin\frac{\beta}{2}\right)^{m'-m} \times {}_2F_1(m'-l;-m-l;m'-m+1;-\tan^2\frac{\beta}{2}) \tag{16}$$

The hypergeometric function is defined to be

$${}_2F_1(a,b;c;z) \equiv 1 + \frac{ab}{c}z + \frac{1}{2}\frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \ldots \tag{17}$$

This equation can generate all the rotation matrices needed by use of some of the following identities:

$$d^l_{m'm}(\cos\beta) = d^l_{mm'}(-\cos\beta) \tag{18}$$

$$\mathcal{D}^l_{m'm}(\mathcal{R}) = (-1)^l\mathcal{D}^l_{m'm}(\bar{\mathcal{R}}), \tag{19}$$

where $\mathcal{R} \equiv$ (inversion) $\times\bar{\mathcal{R}}$.

We can determine the Euler angles α, β, and γ for a given rotation matrix $\mathcal{R}$ from the form of the nine components of the rotation matrix given above.

Therefore, given the rotation matrix $\mathcal{R}$, we can determine the Euler angles using

$$\cos\beta = \mathcal{R}_{zz} \tag{20}$$

$$\sin\beta = \sqrt{1-\mathcal{R}^2_{zz}} \tag{21}$$

If $\sin\beta \neq 0$, then

$$e^{-i\alpha} = \frac{\mathcal{R}_{zx} - i\mathcal{R}_{zy}}{\sin\beta} \tag{22}$$

and

$$e^{-i\gamma} = \frac{\mathcal{R}_{xz} + i\mathcal{R}_{yz}}{-\sin\beta}. \tag{23}$$

If $\sin\beta = 0$, then we can choose $\gamma = 0$, and

$$e^{-i\alpha} = \frac{\mathcal{R}_{xx} - i\mathcal{R}_{xy}}{\mathcal{R}_{zz}} \tag{24}$$

When there is inversion symmetry, we can find α, β, and γ for $\bar{\mathcal{R}}$ and then use Eq. 19 to determine the complete transformation of $\mathcal{R}$.