January 10, 2011

Notes for Lecture #1

1 Introduction

1. Textbook and course structure
2. Motivation
3. Chapters I and 1 and Appendix of Jackson
   (a) Units - SI vs Gaussian
   (b) Laplace and Poisson Equations
   (c) Green's Theorm

2 Units - SI vs Gaussian

Coulomb's law has the form:

$$F = K_C \frac{q_1 q_2}{r_{12}^2}. \tag{1}$$

Ampere's law has the form:

$$F = K_A \frac{i_1 i_2}{r_{12}^2}\, d\mathbf{s_1} \times d\mathbf{s_2} \times \hat{\mathbf{r}}_{\mathbf{12}}, \tag{2}$$

where the current and charge are related by $i_1 = dq_1/dt$ for all unit systems. The two constants K_C and K_A are related so that their ratio K_C/K_A has the units of $(m/s)^2$ and it is *experimentally* known that the ratio has the value $K_C/K_A = c^2$, where c is the speed of light.

The choices for these constants in the SI and Gaussian units are given below:

	CGS (Gaussian)	SI
K_C	1	$\frac{1}{4\pi\epsilon_0}$
K_A	$\frac{1}{c^2}$	$\frac{\mu_0}{4\pi}$

Here, $\frac{\mu_0}{4\pi} \equiv 10^{-7} N/A^2$ and $\frac{1}{4\pi\epsilon_0} = c^2 \cdot 10^{-7} N/A^2 = 8.98755 \times 10^9 N \cdot m^2/C^2$.

Below is a table comparing SI and Gaussian unit systems. The fundamental units for each system are so labeled and are used to define the derived units.

Variable	SI		Gaussian		SI/Gaussian
	Unit	Relation	Unit	Relation	
length	m	fundamental	cm	fundamental	100
mass	kg	fundamental	gm	fundamental	1000
time	s	fundamental	s	fundamental	1
force	N	$kg \cdot m^2/s$	$dyne$	$gm \cdot cm^2/s$	10^5
current	A	fundamental	$statampere$	$statcoulomb/s$	$\frac{1}{10c}$
charge	C	$A \cdot s$	$statcoulomb$	$\sqrt{dyne \cdot cm^2}$	$\frac{1}{10c}$

One advantage of the Gaussian system is that the field vectors: $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{P}, \mathbf{M}$ all have the same physical dimensions., In vacuum, the following equalities hold: $\mathbf{B} = \mathbf{H}$ and $\mathbf{E} = \mathbf{D}$. Also, in the Gaussian system, the dielectric and permittivity constants ϵ and μ are dimensionless.

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$	$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

"Proof" of the identity (Eq. (1.31))

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Noting that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \delta^3(\mathbf{r} - \mathbf{r}') f(\mathbf{r}) = f(\mathbf{r}'), \tag{4}$$

we see that we must show that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) = -4\pi f(\mathbf{r}'). \tag{5}$$

We introduce a small radius a such that:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \lim_{a \to 0} \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}}. \tag{6}$$

For a fixed value of a,

$$\nabla^2 \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}} = \frac{-3a^2}{(|\mathbf{r} - \mathbf{r}'|^2 + a^2)^{5/2}}. \tag{7}$$

If the function $f(\mathbf{r})$ is continuous, we can make a Taylor expansion of it about the point $\mathbf{r} = \mathbf{r}'$, keeping only the first term. The integral over the small sphere about $\mathbf{r}'$ can be carried out analytically, by changing to a coordinate system centered at $\mathbf{r}'$;

$$\mathbf{u} = \mathbf{r} - \mathbf{r}', \tag{8}$$

so that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) \approx f(\mathbf{r}') \lim_{a \to 0} \int_{u<R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}}. \tag{9}$$

We note that

$$\int_{u<R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}} = 4\pi \int_0^R du \; \frac{-3a^2 u^2}{(u^2 + a^2)^{5/2}} = 4\pi \frac{-R^3}{(R^2 + a^2)^{3/2}}. \tag{10}$$

If the infinitesimal value a is $a \ll R$, then $(R^2 + a^2)^{3/2} \approx R^3$ and the right hand side of Eq. 10 is -4π. Therefore, Eq. 9 becomes,

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) \approx f(\mathbf{r}')(-4\pi), \tag{11}$$

which is consistent with Eq. 5 above.