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# Notes for Lecture #10

## Dipole fields

The dipole moment is defined by

$$\mathbf{p} = \int d^3r \rho(r)\mathbf{r}, \tag{1}$$

with the corresponding potential

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \tag{2}$$

and electrostatic field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\,\delta^3(\mathbf{r})\right\}. \tag{3}$$

The last term of the field expression follows from the following derivation. We note that Eq. (3) is poorly defined as $r \to 0$, and consider the value of a small integral of $\mathbf{E}(\mathbf{r})$ about zero. (For this purpose, we are supposing that the dipole $\mathbf{p}$ is located at $\mathbf{r} = \mathbf{0}$.) In this case we will approximate

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r}) d^3r\right) \delta^3(\mathbf{r}). \tag{4}$$

First we note that

$$\int_{r\leq R} \mathbf{E}(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega}. \tag{5}$$

This result follows from the Divergence theorm:

$$\int_{\text{vol}} \nabla\cdot\mathcal{V} d^3r = \int_{\text{surface}} \mathcal{V}\cdot d\mathbf{A}. \tag{6}$$

In our case, this theorem can be used to prove Eq. (5) for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\Phi(\mathbf{r})$ for the $x-$ component for example:

$$\int_{r\leq R} \nabla\Phi(\mathbf{r}) d^3r = \hat{\mathbf{x}}\int_{r\leq R} \nabla\cdot(\hat{\mathbf{x}}\Phi) d^3r + \hat{\mathbf{y}}\int_{r\leq R} \nabla\cdot(\hat{\mathbf{y}}\Phi) d^3r + \hat{\mathbf{z}}\int_{r\leq R} \nabla\cdot(\hat{\mathbf{z}}\Phi) d^3r, \tag{7}$$

which is equal to

$$\int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R} \Phi(\mathbf{r}) R^2 d\Omega\hat{\mathbf{r}}. \tag{8}$$

Thus,

$$\int_{r\leq R} \mathbf{E}(\mathbf{r}) d^3r = -\int_{r\leq R} \nabla\Phi(\mathbf{r}) d^3r = -R^2 \int_{r=R} \Phi(\mathbf{r})\hat{\mathbf{r}} d\mathbf{\Omega}. \tag{9}$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r' \rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\hat{\mathbf{r}})Y_{lm}(\hat{\mathbf{r}'}). \tag{10}$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{-\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}}\right) \end{cases} \tag{11}$$

Therefore, when we evaluate the integral over solid angle $\Omega$ in Eq. (5), only the $l = 1$ term contributes and the effect of the integration reduced to the expression:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\int d^3r'\rho(\mathbf{r}')\frac{r_<}{r_>^2}\hat{\mathbf{r}'}. \tag{12}$$

The choice of $r_<$ and $r_>$ is a choice between the integration variable $r'$ and the sphere radius $R$. If the sphere encloses the charge distribution $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that Eq. (12) becomes

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}\frac{1}{R^2}\int d^3r'\rho(\mathbf{r}')r'\hat{\mathbf{r}'} \equiv -\frac{\mathbf{p}}{3\varepsilon_0}. \tag{13}$$

If the charge distribution $\rho(\mathbf{r}')$ lies outside of the sphere, then $r_> = r'$ and $r_< = R$ so that Eq. (12) becomes

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}d\boldsymbol{\Omega} = -\frac{1}{4\pi\epsilon_0}\frac{4\pi R^2}{3}R\int d^3r'\frac{\rho(\mathbf{r}')}{r'^2}\hat{\mathbf{r}'} \equiv \frac{4\pi R^3}{3}\mathbf{E}(0). \tag{14}$$