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Notes for Lecture #27

Derivation of the Liénard-Wiechert potentials and fields – (first presented in Lecture 17)

When we previously considered solutions to the inhomogeneous electromagnetic wave equations in the Lorentz gauge, (chapter 6 in *Jackson*, we were using MKS units. We keep these units in the following derivations. Consider a point charge q moving on a trajectory $R_q(t)$. We can write its charge density as

$$\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t)), \tag{1}$$

and the current density as

$$\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t)), \tag{2}$$

where

$$\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}. \tag{3}$$

Evaluating the scalar and vector potentials in the Lorentz gauge,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int\int \, d^3r' \, dt' \, \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \, \delta\left(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)\right), \tag{4}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int\int \, d^3r' \, dt' \, \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \, \delta\left(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)\right). \tag{5}$$

We performing the integrations over first d^3r' and then dt', and make use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt' \, f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}, \tag{6}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}. \tag{7}$$

We find

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}, \tag{8}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}}, \tag{9}$$

where we have used the shorthand notation $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ and $\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$.

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \tag{10}$$

and

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t). \tag{11}$$

The trick of evaluating these derivatives is that the retarded time (7) depends on position $\mathbf{r}$ and on itself. We can show the following results using the shorthand notation defined above:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}, \tag{12}$$

and

$$\frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}. \tag{13}$$

Evaluating the gradient of the scalar potential, we find:

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1 - \frac{v^2}{c^2}\right) - \frac{\mathbf{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right], \tag{14}$$

and

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right]. \tag{15}$$

These results can be combined to determine the electric field:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \tag{16}$$

We can also evaluate the curl of $\mathbf{A}$ to find the magnetic field:

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \tag{17}$$

One can show that the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}. \tag{18}$$

Two formulations of electromagnetic fields produced by a charged particle moving at constant velocity

In Chapter 11 of **Jackson** (page 559 – Eqs. 11.151-2 and Fig. 11.8), we derived the electric and magnetic field of a particle having charge q moving at velocity v along the $\hat{\mathbf{x}}_1$ axis. The results are for the fields at the point $\mathbf{r} = b\hat{\mathbf{x}}_2$ are:

$$\mathbf{E}(x_1,x_2,x_3,t) = \mathbf{E}(0,b,0,t) = q\frac{-v\gamma t\hat{\mathbf{x}}_1 + \gamma b\hat{\mathbf{x}}_2}{\left(b^2 + (v\gamma t)^2\right)^{3/2}} \tag{19}$$

and

$$\mathbf{B}(x_1, x_2, x_3, t) = \mathbf{B}(0, b, 0, t) = q\frac{\gamma\beta b\hat{\mathbf{x}}_\mathbf{3}}{(b^2 + (v\gamma t)^2)^{3/2}} \tag{20}$$

for the electric and magnetic fields respectively. The denominators of these expressions are easily interpreted as the distance of the particle from the field point, as measured in the particle's own reference frame. On the other hand, we can consider the same physical problem from the point of view of Liénard-Wiechert potentials:

Consider the electric field produced by a point charge q moving on a trajectory described by $\mathbf{r_0(t)}$ with $\rho(r,t) \equiv q\delta^3(\mathbf{r}-\mathbf{r_0}(t))$. Assume that $\mathbf{v_0}(t) \equiv \partial\mathbf{r_0}(t)/\partial t$ and $\partial^2\mathbf{r_0}(t)/\partial t^2 = 0$. Using the previously derived results for the Liénard Wiechert potentials, changed into Gaussian units, the electric field can be written in the form:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{(1 - v_0^2/c^2)(\mathbf{R} - \mathbf{v_0}R/c)}{(R - \mathbf{v_0}\cdot\mathbf{R}/c)^3} \underset{\text{Gaussian units}}{\longrightarrow} q\frac{(1 - v_0^2/c^2)(\mathbf{R} - \mathbf{v_0}R/c)}{(R - \mathbf{v_0}\cdot\mathbf{R}/c)^3}, \tag{21}$$

where $R \equiv |\mathbf{R}(t_r)|$, $\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{r_0}(t_r)$, and where all quantities which depend on time on the right hand side of the equation are evaluated at the retarded time $t_r \equiv t - R(t_r)/c$. In Gaussian units, the corresponding magnetic field is given by

$$\mathbf{B} = \frac{\mathbf{R}\times\mathbf{E}}{R}. \tag{22}$$

If we evaluate this result for the same case as above (Fig. 11.8 of **Jackson**), $\mathbf{v_0} \equiv v\hat{\mathbf{x}}_\mathbf{1}$, and $\mathbf{R}(t_r) = -vt_r\hat{\mathbf{x}}_\mathbf{1} + b\hat{\mathbf{x}}_\mathbf{2}$. In order to relate this result to Eqs. 19 and 20 above, we need to express t_r in terms of the known quantities. Noting that

$$R(t_r) = c(t - t_r) = \sqrt{(vt_r)^2 + b^2}, \tag{23}$$

we find that t_r must be a solution to the quadratic equation:

$$t_r^2 - 2\gamma^2 t t_r + \gamma^2 t^2 - \gamma^2 b^2/c^2 = 0 \tag{24}$$

with the physical solution:

$$t_r = \gamma\left(\gamma t - \frac{\sqrt{(v\gamma t)^2 + b^2}}{c}\right). \tag{25}$$

Now we can express the length parameter which appears in Eq. 21 as

$$R = \gamma\left(-\beta v\gamma t + \sqrt{(v\gamma t)^2 + b^2}\right). \tag{26}$$

We also can show that the numerator of Eq. 21 can be evaluated:

$$\mathbf{R} - \mathbf{v_0}R/c = -vt\hat{\mathbf{x}}_\mathbf{1} + b\hat{\mathbf{x}}_\mathbf{2}, \tag{27}$$

and the denominator can be evaluated:

$$R - \mathbf{v_0}\cdot\mathbf{R}/c = \frac{\sqrt{(v\gamma t)^2 + b^2}}{\gamma}. \tag{28}$$

Substituting these results into Eqs. 21 and 22, we obtain the same electric and magnetic fields as given in Eqs. 19 and 20 from the field transformation approach.