January 18, 2011

Notes for Lecture #3

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{1}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0. \tag{2}$$

In fact, the Laplace equation is the "homogeneous" version of the Poisson equation. The Green's function allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot \hat{\mathbf{r}}'. \tag{3}$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation $\Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r})$, for any constant C. In lecture notes #2, we discussed one method of constructing Green's functions that works for one-dimensional systems. Below, we discuss another method that is generalizable for higher dimensional systems.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x)u_m(x)\, dx = \delta_{nm}. \tag{4}$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x)u_n(x') = \delta(x-x'). \tag{5}$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x,x') = -4\pi\delta(x-x'). \tag{6}$$

Therefore, if

$$\frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x), \tag{7}$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write the Green's functions as

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}. \tag{8}$$

For example, consider the example discussed in Lecture #2 in the interval $-a \le x \le a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{9}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{10}$$

The Green's function for this case as:

$$G(x,x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)\sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{11}$$

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \le x \le a$ from the integral

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} dx'\, G(x,x')\rho(x'),. \tag{12}$$

The boundary corrected full solution within the interval $-a \le x \le a$ is given by

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0} \left(16 \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2} \right). \tag{13}$$

The orthogonal function expansion method can easily be extended to two and three dimensions. For example if $\{u_n(x)\}$, $\{v_n(x)\}$, and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written:

$$G(x,x',y,y',z,z') = 4\pi \sum_{lmn} \frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n}, \tag{14}$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y), \quad \text{and} \quad \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z). \tag{15}$$

See Eq. 3.167 in **Jackson** for an example.

As discussed in Lecture Notes #2, an alternative method of finding Green's functions for second order ordinary differential equations is based on a product of two independent solutions of the homogeneous equation, $u_1(x)$ and $u_2(x)$, which satisfy the boundary conditions at x_1 and x_1, respectively:

$$G(x,x') = Ku_1(x_<)u_2(x_>), \quad \text{where} \quad K \equiv \frac{4\pi}{\frac{du_1}{dx}u_2 - u_1\frac{du_2}{dx}}, \tag{16}$$

with $x_<$ meaning the smaller of x and x' and $x_>$ meaning the larger of x and x'. For example, we have previously discussed the example of the one dimensional Poisson equation to have the form:

$$G(x,x') = 4\pi x_<. \tag{17}$$

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of **Jackson**. For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y'). \tag{18}$$

If the functions $\{u_n(x)\}$ satisfy Eq. 7, then we must require that G satisfy the equation:

$$\nabla^2 G = \sum_n u_n(x)u_n(x')\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(x-x')\delta(y-y'). \tag{19}$$

The $y-$dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'), \tag{20}$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0, \tag{21}$$

and a constant related to the Wronskian:

$$K_n \equiv \frac{4\pi}{\frac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\frac{dv_{n_2}}{dy}}. \tag{22}$$

If these functions also satisfy the appropriate boundary conditions, we can then construct the 2-dimensional Green's function from

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')K_n v_{n_1}(y_<)v_{n_2}(y_>). \tag{23}$$

For example, a Green's function for a two-dimensional rectangular system with $0 \leq x \leq a$ and $0 \leq y \leq b$, which vanishes on each of the boundaries can be expanded:

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}. \tag{24}$$

As an example, we can use this result to solve the 2-dimensional Laplace equation in the square region $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary condition $\Phi(x,0) = \Phi(0,y) = \Phi(1,y) = 0$ and $\Phi(x,1) = V_0$. In this case, in determining $\Phi(x,y)$ using Eq. (3) there is no volume contribution (since the charge is zero) and the "surface" integral becomes a line integral $0 \leq x' \leq 1$ for $y' = 1$. Using the form from Eq. (24) with $a = b = 1$, it can be shown that the result takes the form:

$$\Phi(x,y) = \sum_{n=0}^{\infty} 4V_0 \quad \frac{\sin[(2n+1)\pi x]\sinh[(2n+1)\pi y]}{(2n+1)\pi\sinh[(2n+1)\pi]} \tag{25}$$