Notes for Lecture #30

Synchrotron Radiation

For this analysis we will use the geometry shown in Fig. 14.9 of *Jackson*. A particle with charge \(q\) is moving in a circular trajectory with radius \(\rho\) and speed \(v\). Its trajectory as a function of time \(t\) is given by

\[
R_q(t) = \rho \sin(vt/\rho) \hat{x} + \rho (1 - \cos(vt/\rho)) \hat{y}.
\]

(1)

Its velocity as a function of time is given by

\[
v_q(t) = v \cos(vt/\rho) \hat{x} + v \sin(vt/\rho) \hat{y}.
\]

(2)

The spectral intensity that we must evaluate is given by the expression (Eq. 14.67 in Jackson)

\[
\frac{d^2I}{d\omega d\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \beta) e^{i\omega(t - \mathbf{\hat{r}} \cdot \mathbf{R}_q(t)/c)} dt \right|^2.
\]

(3)

After some algebra, this expression can be put into the form

\[
\frac{d^2I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{4\pi^2 c} \left\{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \right\},
\]

(4)

where the amplitude for the light polarized along the \(y\)-axis is given by

\[
C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho)e^{i\omega(t - \rho \cos \theta \sin(vt/\rho))}
\]

(5)

and the amplitude for the light polarized perpendicular to \(\mathbf{\hat{y}}\) and \(\mathbf{\hat{r}}\) is given by

\[
C_{\perp}(\omega) \int_{-\infty}^{\infty} dt \sin \theta \cos(vt/\rho)e^{i\omega(t - \rho \cos \theta \sin(vt/\rho))}.
\]

(6)

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light \((v \approx c(1 - 1/(2\gamma^2))\) passing a beam line port. In addition, because of the design of the radiation ports, \(\theta \approx 0\), and the relevant integration times \(t\) are close to \(t \approx 0\). This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency

\[
\omega_c \equiv \frac{3c\gamma^3}{2\rho}.
\]

(7)

The resultant intensity is then given by

\[
\frac{d^2I}{d\omega d\Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left( \frac{\omega}{\omega_c} \right)^2 (1 + \gamma^2 \theta^2)^2 \left\{ \left[ K_{2/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[ K_{1/3} \left( \frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 \right\}.
\]

(8)
By plotting this expression as a function of $\omega$, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of $\omega/\omega_c$ for $\gamma\theta = 0, 0.5$ and 1.

The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of $v/\rho$. For this case we need to reconsider Eqs. 5 and 6. There is a very convenient Bessel function identity of the form:

$$e^{-ia\sin \alpha} = \sum_{m=-\infty}^{\infty} J_m(a)e^{-im\alpha}.$$  \hfill (9)

Here $J_m(a)$ is a Bessel function of integer order $m$. In our case $a = \frac{\omega c}{v} \cos \theta$ and $\alpha = \frac{vt}{\rho}$. Analyzing the “parallel” component we have

$$C_\parallel = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \int_{-\infty}^{\infty} dt e^{i\omega(t - \frac{\xi}{\rho} \cos \theta \sin(vt/\rho))} = \frac{c}{-i\omega \rho} \frac{\partial}{\partial \cos \theta} \sum_{m=-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) 2\pi \delta(\omega - m\frac{v}{\rho}).$$ \hfill (10)

In determining this result, we have used the identity

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi \delta(\omega - m\frac{v}{\rho}).$$ \hfill (11)

Eq. 10 can be simplified to show that

$$C_\parallel = 2\pi i \sum_{m=-\infty}^{\infty} J'_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m\frac{v}{\rho}),$$ \hfill (12)

where $J'_m(a) \equiv \frac{dJ_m(a)}{da}$. The “perpendicular” component can be analyzed in a similar way, using integration by parts to eliminate the extra $\cos(vt/\rho)$ term in the argument. The result is

$$C_\perp = 2\pi \frac{\tan \theta}{v/c} \sum_{m=-\infty}^{\infty} J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \delta(\omega - m\frac{v}{\rho}).$$ \hfill (13)
In both of these expressions, the sum over \( m \) includes both negative and positive values of \( m \). However, only the positive values of \( \omega \) and therefore positive values of \( m \) are of interest, and if we needed to use the negative \( m \) values, we could use the identity

\[
J_{-m}(a) = (-1)^m J_m(a).
\]  

Combining these results, we find that the intensity spectrum for this case consists of a series of discrete frequencies which are multiples of \( v/\rho \).

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{c} \sum_{m=0}^{\infty} \delta(\omega - m \frac{v}{\rho}) \left\{ \left[ J'_m \left( \frac{\omega \rho}{c} \cos \theta \right) \right]^2 + \frac{\tan^2 \theta}{v^2/c^2} \left[ J_m \left( \frac{\omega \rho}{c} \cos \theta \right) \right]^2 \right\}.
\]  

These results were derived by Julian Schwinger (Phys. Rev. 75, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson’s text. It should have some implications for Astronomical observations, but I have not yet found any references for that. For information on man-made synchrotron sources, the following web page is useful: http://www.als.lbl.gov/als/synchrotron_sources.html.