PHY 341/641
Thermodynamics and Statistical Physics

Lecture 10

Probability concepts (Chapter 3 in STP)

A. Binomial distribution (continued)
B. Central Limit theorem
C. Poisson distribution and others

Comment on HW problem:

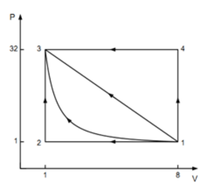


Figure 2.17: Illustration of various thermodynamic processes discussed in Problem 2.56. of the pressure P and the volume V are Pa and m^3, respectively.

For analyzing work $W_{1\to 3}$ (along straight line):

$$P = P(V) = P_1 + \frac{P_3 - P_1}{V_3 - V_1}\left(V - V_1\right) \quad \text{for } V_3 \le V \le V_1$$

$$W_{1\to 3} = -\int_{V_1}^{V_3} P(V)dV = -\int_{V_1}^{V_3}\left(P_1 + \frac{P_3 - P_1}{V_3 - V_1}\left(V - V_1\right)\right)dV$$

Binomial distribution (review) –

Assume each elemental event has 2 possible outcomes (for example H with probability p and T with probability q=1-p). $P_N(n)$ gives the probability distribution for N elemental events having n instances of H.

$$P_N(n) = \frac{N!}{n!\left(N-n\right)!} p^n q^{N-n}$$

Average value:

$$\langle n \rangle = pN$$

Variance:

$$\langle n^2 \rangle - \langle n \rangle^2 \equiv \sigma^2 = Npq$$

$$P_{N\to\infty}(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(n - \langle n \rangle\right)^2 / 2\sigma^2}$$

Check :

$$\langle n\rangle = \int_{-\infty}^{\infty} nP_{\infty}(n)dn = \int_{-\infty}^{\infty} n\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\left(n-\langle n\rangle\right)^2/2\sigma^2}dn$$

$$\sigma^2 = \left\langle n^2\right\rangle - \langle n\rangle^2 = \int_{-\infty}^{\infty} n^2\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\left(n-\langle n\rangle\right)^2/2\sigma^2}dn - \langle n\rangle^2$$

More general importance of Gaussian probability distribution – "central limit theorem"

One-dimensional random walk

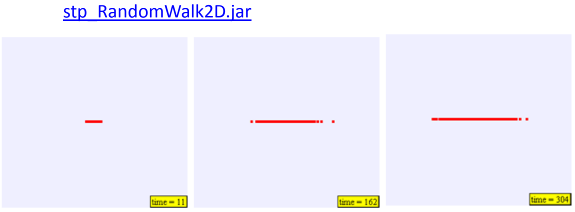
- It is possible to describe a version of a random walk process as a binomial distribution. If each step has a fixed length s, then an elementary event is a step to the right with probability p or a step to the left with probability q. The probability of n steps to the right is then the binomial distribution.

$$P_N(n) = \frac{N!}{n!\left(N-n\right)!}p^n q^{N-n}$$

$$P_{N\to\infty}(n) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\left(n-\langle n\rangle\right)^2/2\sigma^2}$$

Example of random walk:

stp_RandomWalk2D.jar



One-dimensional fixed step walk, continued

For N total steps and n steps to the right with probability $P_N(n)$, displacement is

$x=ns-(N-n)s=(2n-N)s$

$\langle x\rangle=(2\langle n\rangle-N)s=(2p-1)Ns$

$\langle x^2\rangle-\langle x\rangle^2=4pqNs^2$

Generalization – one-dimensional variable step (s_i) walk (derivations following *Fundamentals of statistical and thermal physics* by F. Reif (1965))

$$x=\sum_{i=1}^{N} s_i$$

Let $w(s_i)ds_i$ denote the probability that i^{th} displacement is between s_i and s_i+ds_i

The probability distribution for displacement x is

$$P(x)dx=\iiint\ldots\iint ds_1 ds_2\ldots ds_N w(s_1)w(s_2)\cdots w(s_N)$$

where $x<\sum_{i=1}^{N} s_i<x+dx$

$$P(x)dx=\iiint\ldots\iint ds_1 ds_2\ldots ds_N w(s_1)w(s_2)\cdots w(s_N)\left[\delta\left(x-\sum_{i=1}^{N} s_i\right)\right]dx$$

Evaluation:

$$\delta(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{iku} \qquad \delta(x-\sum s_i)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik\left(\sum s_i - x\right)}$$

Let $Q(k)\equiv\int_{-\infty}^{\infty} ds e^{iks} w(s)$

$$P(x)dx=\iiint\ldots\iint ds_1 ds_2\ldots ds_N w(s_1)w(s_2)\cdots w(s_N)\left[\delta\left(x-\sum_{i=1}^{N} s_i\right)\right]dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left(Q(k)\right)^N e^{-ikx}dx$$

Evaluation in the limit of $N \to \infty$

$$P(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left(Q(k)\right)^N e^{-ikx}dx$$

$$Q(k) \equiv \int_{-\infty}^{\infty} ds e^{iks} w(s) \approx \int_{-\infty}^{\infty} ds\left(1 + iks - \tfrac{1}{2}k^2 s^2 \cdots\right)w(s) \approx 1 + ik\langle s\rangle - \tfrac{1}{2}k^2\langle s^2\rangle$$

$$\ln\left(Q(k)\right)^N = N\ln\left(Q(k)\right) \approx N\ln\left(1 + ik\langle s\rangle - \tfrac{1}{2}k^2\langle s^2\rangle\right)$$

Assuming $k\langle s\rangle << 1$:

$$N\ln\left(Q(k)\right) \approx N\left(ik\langle s\rangle - \tfrac{1}{2}k^2\left(\langle s^2\rangle - \langle s\rangle^2\right)\right) \equiv N\left(ik\langle s\rangle - \tfrac{1}{2}k^2\langle \Delta s^2\rangle\right)$$

Let $\sigma^2 = N\langle \Delta s^2\rangle$ $\quad\quad$ $\langle x\rangle = N\langle s\rangle$

$$\Rightarrow P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}}$$

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Poisson probability distribution –
Approximation to binomial distribution for small p

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

Assume $N >> n$ $\quad\quad$ $p << 1$

$$\frac{N!}{n!(N-n)!} \approx \frac{N^n}{n!}$$

$$\ln q^{N-n} = (N-n)\ln(1-p) \approx -(N-n)p \approx -Np$$

$$P_N(n) \approx \frac{N^n}{n!} p^n e^{-Np} = \frac{\langle n\rangle^n}{n!} e^{-\langle n\rangle}$$

Check: $\quad \langle n\rangle = \sum_{n=0}^{\infty} n \frac{\langle n\rangle^n}{n!} e^{-\langle n\rangle} = \langle n\rangle$

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Other probability distributions:

Continuous distributions $\quad p(x) \equiv$ probability distribution

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Probability for x in given range: $P(a \le x \le b) = \int_a^b p(x)dx$

Average value of any function for $f(x)$: $\langle f\rangle = \int_{-\infty}^{\infty} f(x)p(x)dx$

Example:

$$p(x) = \frac{1}{\pi}\frac{\gamma}{(x-a)^2 + \gamma^2}$$

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