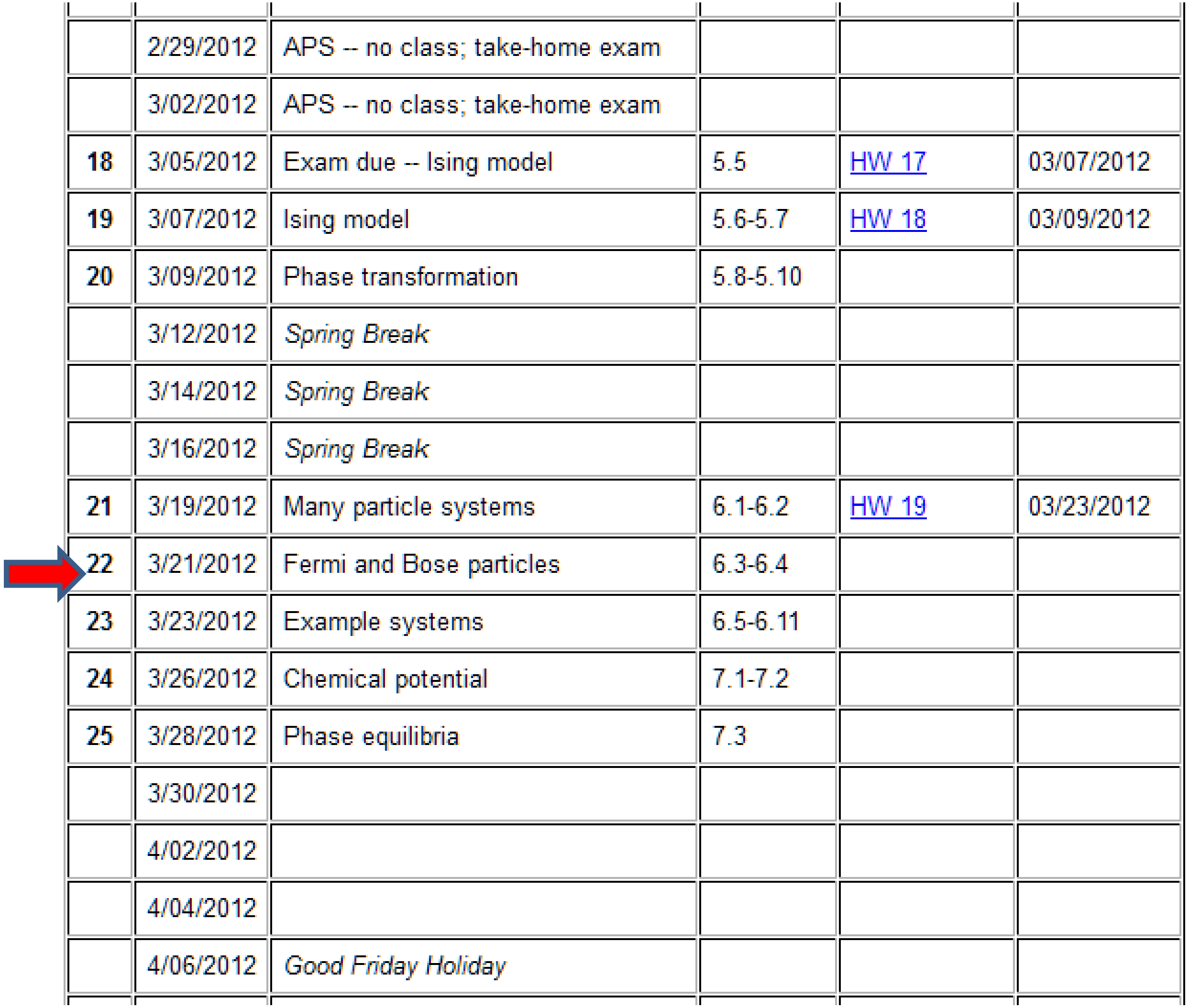
PHY 341/641
Thermodynamics and Statistical Physics

Lecture 22

Many particle systems (Chapter 6 in STP)

- Maxwell velocity distribution
- Non-interacting Fermi particles
- Non-interacting Bose particles



	2/29/2012	APS -- no class; take-home exam			
	3/02/2012	APS -- no class; take-home exam			
18	3/05/2012	Exam due -- Ising model	5.5	HW 17	03/07/2012
19	3/07/2012	Ising model	5.6-5.7	HW 18	03/09/2012
20	3/09/2012	Phase transformation	5.8-5.10		
	3/12/2012	*Spring Break*			
	3/14/2012	*Spring Break*			
	3/16/2012	*Spring Break*			
21	3/19/2012	Many particle systems	6.1-6.2	HW 19	03/23/2012
22	3/21/2012	Fermi and Bose particles	6.3-6.4		
23	3/23/2012	Example systems	6.5-6.11		
24	3/26/2012	Chemical potential	7.1-7.2		
25	3/28/2012	Phase equilibria	7.3		
	3/30/2012				
	4/02/2012				
	4/04/2012				
	4/06/2012	*Good Friday Holiday*			

Reminder – second exam in April

-- student presentations 4/30, 5/2 (need to pick topics)

Interacting systems – N identical particles of mass m: Classical treatment

$$Z(T,V,N) = \frac{1}{N!h^{3N}} \int d^{3N}r \int d^{3N}p \, e^{-\frac{\beta}{2m}\left(\sum_i \mathbf{p}_i^2\right) - \beta V(\{\mathbf{r}_i\})}$$

$$Z(T,V,N) = \frac{1}{N!h^{3N}} \int d^{3N}r e^{-\beta V(\{\mathbf{r}_i\})} \int d^{3N}p \, e^{-\frac{\beta}{2m}\left(\sum_i \mathbf{p}_i^2\right)}$$

$$\mathbf{p}_i = m\mathbf{v}_i \qquad\qquad d^3p = 4\pi m^3 v^2 dv$$

$$\int d^{3N}p \, e^{-\frac{\beta}{2m}\left(\sum_i \mathbf{p}_i^2\right)} = \left(4\pi m^3 \int v^2 dv e^{-\beta m v^2/2}\right)^N$$

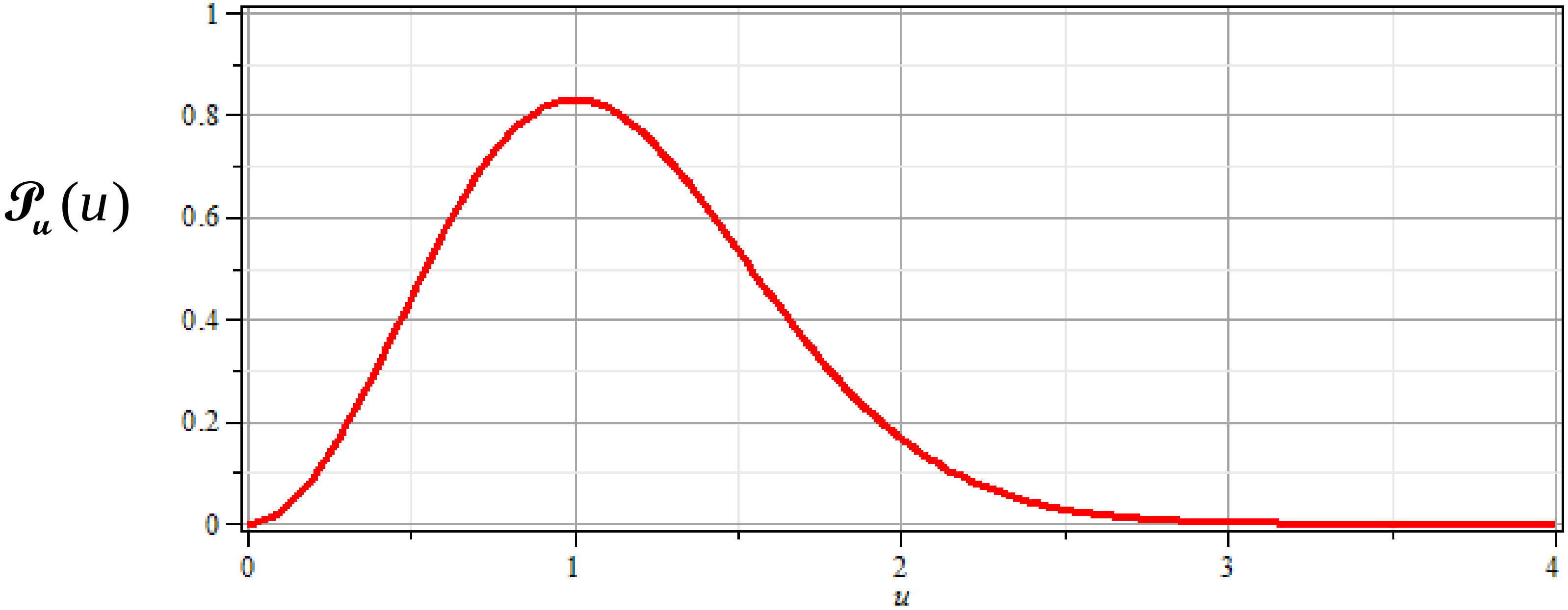
Probability of finding particle of velocity between v and $v + dv$:

$$\mathscr{P}(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

Maxwell velocity distribution

$$\mathcal{P}(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

$$\mathcal{P}(v)dv = \mathcal{P}_u(u)du = \frac{4}{\sqrt{\pi}}u^2 e^{-u^2}du \quad \text{where } u \equiv \sqrt{\frac{m}{2kT}}v$$



➔ For classical particles the Maxwell velocity distribution is the same for all particle interaction potentials.

Statistics of non-interacting quantum particles

Single particle states: ε_k

Single particle occupation numbers: n_k

Bose particles (integer spin): $n_k = 0,1,2,3,\cdots$

Fermi particles ($\frac{1}{2}$ integer spin): $n_k = 0,1$

Grand partition function for these systems:

$$Z_G(T,\mu) = \sum_s e^{-\beta\left(E_s - \mu N_s\right)} \text{ summing over all microstates } s$$

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$Z_G(T,\mu) = \sum_s e^{-\beta\left(E_s - \mu N_s\right)}$ summing over all microstates s

$$E_s = \sum_k n_k^s \varepsilon_k \qquad\qquad N_s = \sum_k n_k^s$$

$$Z_G(T,\mu) = \prod_k \left(\sum_s e^{-\beta\left(n_k^s \varepsilon_k - \mu n_k^s\right)} \right)$$

$$\equiv \prod_k Z_{G,k}(T,\mu)$$

where $Z_{G,k}(T,\mu) \equiv \sum_s e^{-\beta\left(n_k^s \varepsilon_k - \mu n_k^s\right)}$

Fermi particle case : $n_k^s = 0,1$

$$Z_{G,k}(T,\mu) \equiv \sum_s e^{-\beta\left(n_k^s \varepsilon_k - \mu n_k^s\right)}$$

$$= 1 + e^{-\beta(\varepsilon_k - \mu)}$$

Landau potential for this case :

$$\Omega_\mathrm{k} = -kT \ln Z_{G,k} = -kT \ln\left(1 + e^{-\beta(\varepsilon_k - \mu)}\right)$$

Mean occupancy numbers :

$$\left\langle n_k \right\rangle = -\frac{\partial \Omega_\mathrm{k}}{\partial \mu} = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}$$

Bose particle case : $n_k^s = 0,1,2,3,4,\cdots$

$$Z_{G,k}(T,\mu) \equiv \sum_s e^{-\beta\left(n_k^s \varepsilon_k - \mu n_k^s\right)}$$

$$= \sum_{n_k^s=0}^{\infty} e^{-\beta(\varepsilon_k-\mu)n_k^s} = \frac{1}{1-e^{-\beta(\varepsilon_k-\mu)}}$$

Landau potential for this case :

$$\Omega_k = -kT \ln Z_{G,k} = kT \ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)$$

Mean occupancy numbers :

$$\langle n_k \rangle = -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$$

Bose particle case : $n_k^s = 0,1,2,3,4,\cdots$

Note a detail:

$$Z_{G,k}(T,\mu) = \sum_{n_k^s=0}^{\infty} e^{-\beta(\varepsilon_k - \mu)n_k^s} = \frac{1}{1 - e^{-\beta(\varepsilon_k - \mu)}}$$

Note that the summation of the geometric series

implies that $e^{-\beta(\varepsilon_k - \mu)} < 1$

$\Rightarrow e^{\beta\mu} < 1 \quad \text{or} \quad \mu < 0$

Summary of results for Landau potential for non-interacting Fermi and Bose particles

Fermi particles:

$$\Omega(T,\mu)=\sum_k \Omega_{\mathrm{k}} = -kT\sum_k \ln Z_{G,k} = -kT\sum_k \ln\left(1+e^{-\beta(\varepsilon_k-\mu)}\right)$$

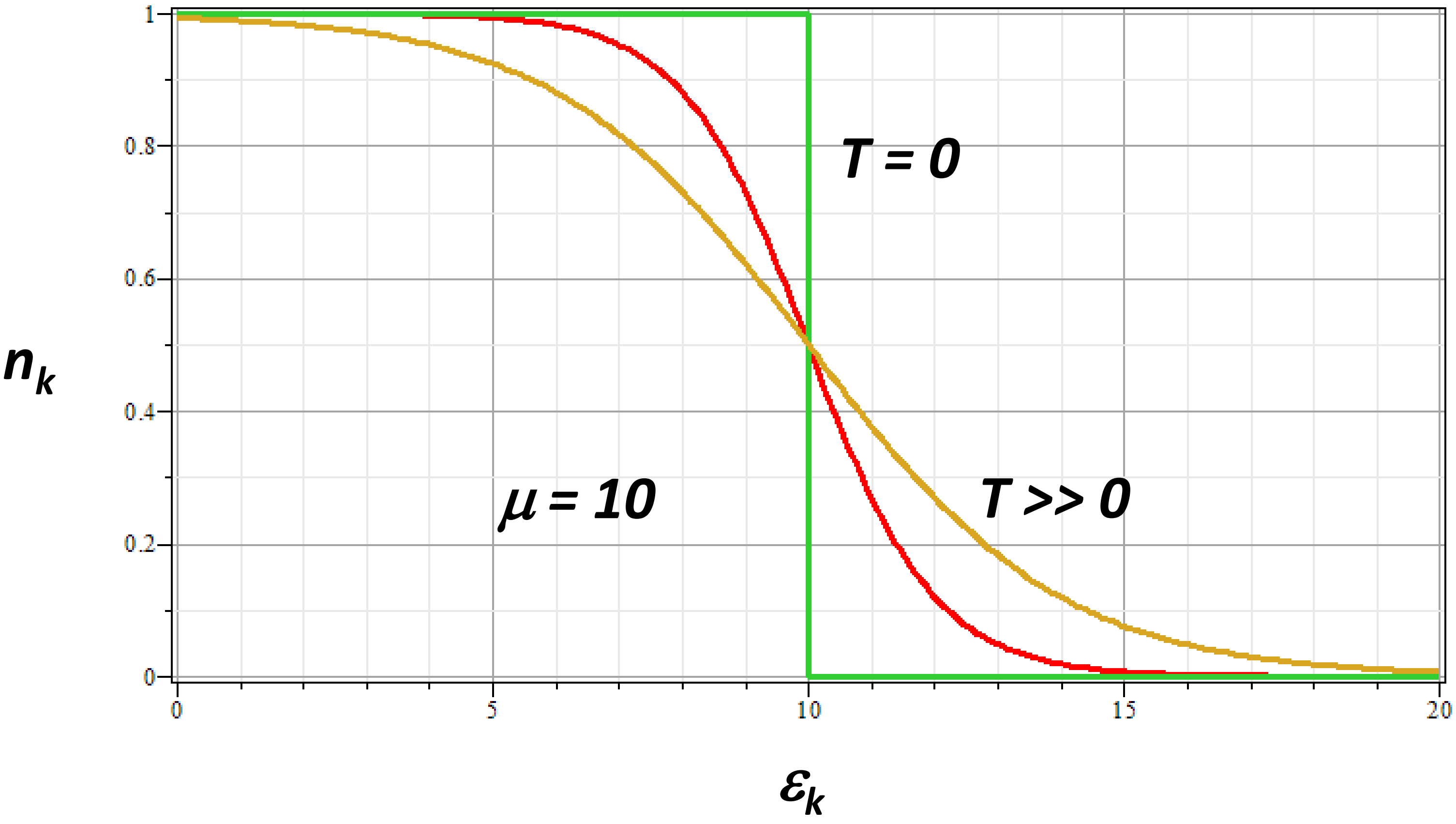
$$\langle n_k\rangle = \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}$$

Bose particles:

$$\Omega(T,\mu)=\sum_k \Omega_{\mathrm{k}} = -kT\sum_k \ln Z_{G,k} = kT\sum_k \ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)$$

$$\langle n_k\rangle = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$$

Case of Fermi particles



Case of Fermi particles

Non-interacting spin ½ particles of mass *m* at *T=0* moving in 3-dimensions in large box of volume *V=L³*:

Assume that each state e_k is doubly occupied (due to spin)

$$N=\sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}$$

$$\varepsilon_{n_x,n_y,n_z}=\frac{h^2\left(n_x^2+n_y^2+n_z^2\right)}{8mL^2} \quad n_x,n_y,n_z=1,2,3\cdots$$

$$\text{In the limit } L\to\infty,\ \varepsilon_{\mathrm{k}} \to \frac{\hbar^2k^2}{2m}$$

$$\sum_k \to 2\left(\frac{L}{2\pi}\right)^3\int d^3k$$

spin degeneracy

Case of Fermi spin ½ particles for *T →0*.

$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} \approx \begin{cases} 1 & \text{for } \varepsilon_k < \mu \\ 0 & \text{for } \varepsilon_k > \mu \end{cases}$$

$$N = \sum_k \langle n_k \rangle \to 2\left(\frac{L}{2\pi}\right)^3 \int_{\varepsilon_k < \mu} d^3k = 2\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} k_F^3$$

$$\mu = \frac{\hbar^2 k_F^2}{2m} \equiv \varepsilon_F$$

$$N = \frac{V}{3\pi^2}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2}$$

$$\Rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$