

PHY 341/641 Thermodynamics and Statistical Physics

Lecture 22

Many particle systems (Chapter 6 in STP)

- Maxwell velocity distribution
- Non-interacting Fermi particles
- Non-interacting Bose particles

3/21/2012

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1

	2/29/2012	APS -- no class, take-home exam			
	3/02/2012	APS -- no class, take-home exam			
18	3/05/2012	Exam due -- Ising model	5.5	HW 17	03/07/2012
19	3/07/2012	Ising model	5.6-5.7	HW 18	03/09/2012
20	3/09/2012	Phase transformation	5.8-5.10		
	3/12/2012	Spring Break			
	3/14/2012	Spring Break			
	3/16/2012	Spring Break			
21	3/19/2012	Many particle systems	6.1-6.2	HW 19	03/23/2012
22	3/21/2012	Fermi and Bose particles	6.3-6.4		
23	3/23/2012	Example systems	6.5-6.11		
24	3/26/2012	Chemical potential	7.1-7.2		
25	3/28/2012	Phase equilibria	7.3		
	3/30/2012				
	4/02/2012				
	4/04/2012				
	4/06/2012	Good Friday Holiday			

Reminder -- second exam in April
-- student presentations 4/30, 5/2 (need to pick topics)

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2

Interacting systems -- N identical particles of mass m:
Classical treatment

$$Z(T, V, N) = \frac{1}{N! h^{3N}} \int d^3N r \int d^3N p e^{-\frac{\beta}{2m} \left(\sum_i p_i^2 \right) - \beta V(\{r_i\})}$$

$$Z(T, V, N) = \frac{1}{N! h^{3N}} \int d^3N r e^{-\beta V(\{r_i\})} \int d^3N p e^{-\frac{\beta}{2m} \left(\sum_i p_i^2 \right)}$$

$$\mathbf{p}_i = m \mathbf{v}_i \quad d^3 p = 4\pi m^3 v^2 dv$$

$$\int d^3N p e^{-\frac{\beta}{2m} \left(\sum_i p_i^2 \right)} = \left(4\pi m^3 \int v^2 dv e^{-\beta m v^2 / 2} \right)^N$$

Probability of finding particle of velocity between v and $v + dv$:

$$\mathcal{P}(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT}$$

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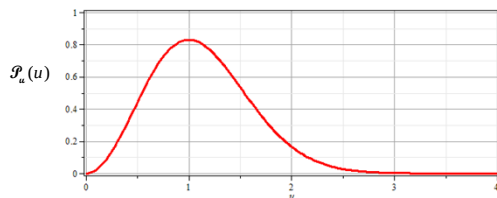
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3

Maxwell velocity distribution

$$\mathcal{P}(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT}$$

$$\mathcal{P}(v)dv = \mathcal{P}_u(u)du = \frac{4}{\sqrt{\pi}} u^2 e^{-u^2} du \quad \text{where } u \equiv \sqrt{\frac{m}{2kT}} v$$



→ For classical particles the Maxwell velocity distribution is the same for all particle interaction potentials.

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4

Statistics of non-interacting quantum particles

Single particle states : $\mathcal{E}_k$

Single particle occupation numbers : n_k

Bose particles (integer spin) : $n_k = 0, 1, 2, 3, \dots$

Fermi particles ($\frac{1}{2}$ integer spin) : $n_k = 0, 1$

Grand partition function for these systems:

$$Z_G(T, \mu) = \sum_s e^{-\beta(E_s - \mu N_s)} \quad \text{summing over all microstates } s$$

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5

Grand partition function for these systems:

$$Z_G(T, \mu) = \sum_s e^{-\beta(E_s - \mu N_s)} \quad \text{summing over all microstates } s$$

$$E_s = \sum_k n_k^s \mathcal{E}_k \quad N_s = \sum_k n_k^s$$

$$Z_G(T, \mu) = \prod_k \left(\sum_s e^{-\beta(n_k^s \mathcal{E}_k - \mu n_k^s)} \right) \\ \equiv \prod_k Z_{G,k}(T, \mu)$$

$$\text{where } Z_{G,k}(T, \mu) \equiv \sum_s e^{-\beta(n_k^s \mathcal{E}_k - \mu n_k^s)}$$

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6

Fermi particle case : $n_k^s = 0, 1$

$$Z_{G,k}(T, \mu) \equiv \sum_s e^{-\beta(n_k^s \epsilon_k - \mu n_k^s)}$$

$$= 1 + e^{-\beta(\epsilon_k - \mu)}$$

Landau potential for this case :

$$\Omega_k = -kT \ln Z_{G,k} = -kT \ln(1 + e^{-\beta(\epsilon_k - \mu)})$$

Mean occupancy numbers :

$$\langle n_k \rangle = -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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7

Bose particle case : $n_k^s = 0, 1, 2, 3, 4, \dots$

$$Z_{G,k}(T, \mu) \equiv \sum_s e^{-\beta(n_k^s \epsilon_k - \mu n_k^s)}$$

$$= \sum_{n_k^s=0}^{\infty} e^{-\beta(\epsilon_k - \mu)n_k^s} = \frac{1}{1 - e^{-\beta(\epsilon_k - \mu)}}$$

Landau potential for this case :

$$\Omega_k = -kT \ln Z_{G,k} = kT \ln(1 - e^{-\beta(\epsilon_k - \mu)})$$

Mean occupancy numbers :

$$\langle n_k \rangle = -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

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8

Bose particle case : $n_k^s = 0, 1, 2, 3, 4, \dots$

Note a detail:

$$Z_{G,k}(T, \mu) = \sum_{n_k^s=0}^{\infty} e^{-\beta(\epsilon_k - \mu)n_k^s} = \frac{1}{1 - e^{-\beta(\epsilon_k - \mu)}}$$

Note that the summation of the geometric series

implies that $e^{-\beta(\epsilon_k - \mu)} < 1$

$$\Rightarrow e^{\beta\mu} < 1 \quad \text{or} \quad \mu < 0$$

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9

Summary of results for Landau potential for non-interacting Fermi and Bose particles

Fermi particles:

$$\Omega(T, \mu) = \sum_k \Omega_k = -kT \sum_k \ln Z_{G,k} = -kT \sum_k \ln(1 + e^{-\beta(\epsilon_k - \mu)})$$

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

Bose particles:

$$\Omega(T, \mu) = \sum_k \Omega_k = -kT \sum_k \ln Z_{G,k} = kT \sum_k \ln(1 - e^{-\beta(\epsilon_k - \mu)})$$

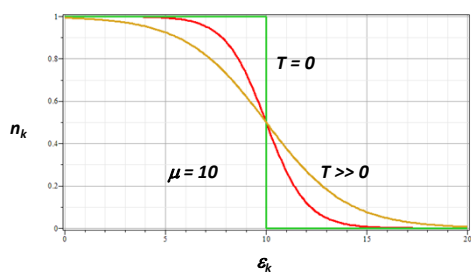
$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

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10

Case of Fermi particles



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11

Case of Fermi particles

Non-interacting spin $\frac{1}{2}$ particles of mass m at $T=0$ moving in 3-dimensions in large box of volume $V=L^3$: Assume that each state ϵ_k is doubly occupied (due to spin)

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$\epsilon_{n_x, n_y, n_z} = \frac{h^2(n_x^2 + n_y^2 + n_z^2)}{8mL^2} \quad n_x, n_y, n_z = 1, 2, 3, \dots$$

$$\text{In the limit } L \rightarrow \infty, \epsilon_k \rightarrow \frac{\hbar^2 k^2}{2m}$$

$$\sum_k \rightarrow 2 \left(\frac{L}{2\pi} \right)^3 \int d^3k$$

spin degeneracy

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12

Case of Fermi spin $\frac{1}{2}$ particles for $T \rightarrow 0$.

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \approx \begin{cases} 1 & \text{for } \epsilon_k < \mu \\ 0 & \text{for } \epsilon_k > \mu \end{cases}$$

$$N = \sum_k \langle n_k \rangle \rightarrow 2 \left(\frac{L}{2\pi} \right)^3 \int_{\epsilon_k < \mu} d^3k = 2 \left(\frac{L}{2\pi} \right)^3 \frac{4\pi}{3} k_F^3$$

$$\mu = \frac{\hbar^2 k_F^2}{2m} \equiv \epsilon_F$$

$$N = \frac{V}{3\pi^2} \left(\frac{2m\epsilon_F}{\hbar^2} \right)^{3/2}$$

$$\Rightarrow \epsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3}$$

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13
