PHY 341/641
Thermodynamics and Statistical Physics

Lecture 23

Many particle systems (Chapter 6 in STP)

- Non-interacting Fermi particles
- Non-interacting Bose particles

17	2/24/2012	Introduction to magnetic systems	5.1-5.5		
	2/27/2012	APS -- no class; take-home exam			
	2/29/2012	APS -- no class; take-home exam			
	3/02/2012	APS -- no class; take-home exam			
18	3/05/2012	Exam due -- Ising model	5.5	HW 17	03/07/2012
19	3/07/2012	Ising model	5.6-5.7	HW 18	03/09/2012
20	3/09/2012	Phase transformation	5.8-5.10		
	3/12/2012	*Spring Break*			
	3/14/2012	*Spring Break*			
	3/16/2012	*Spring Break*			
21	3/19/2012	Many particle systems	6.1-6.2	HW 19	03/23/2012
22	3/21/2012	Fermi and Bose particles	6.3-6.4		
23	3/23/2012	Bose and Fermi particles	6.5-6.11	HW 20	03/28/2012
24	3/26/2012	Bose and Fermi particles	6.5-6.11		
25	3/28/2012	Chemical potential	7.1-7.2		
26	3/30/2012	Phase equilibria	7.3		
	4/02/2012				

Reminder – second exam in April
-- student presentations 4/30, 5/2 (need to pick topics)

Summary of results for Landau potential for non-interacting Fermi and Bose particles

Fermi particles:

$$\Omega(T,\mu) = \sum_k \Omega_k = -kT\sum_k \ln Z_{G,k} = -kT\sum_k \ln\left(1+e^{-\beta(\varepsilon_k-\mu)}\right)$$

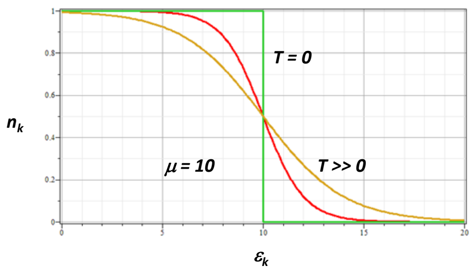
$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}$$

Bose particles:

$$\Omega(T,\mu) = \sum_k \Omega_k = -kT\sum_k \ln Z_{G,k} = kT\sum_k \ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)$$

$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$$

Case of Fermi particles



Case of Fermi particles

Non-interacting spin ½ particles of mass *m* at *T=0* moving in 3-dimensions in large box of volume $V=L^3$:

Assume that each state e_k is doubly occupied (due to spin)

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}$$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2\left(n_x^2 + n_y^2 + n_z^2\right)}{8mL^2} \quad n_x, n_y, n_z = 1,2,3\cdots$$

$$\text{In the limit } L \to \infty,\ \varepsilon_k \to \frac{\hbar^2 k^2}{2m}$$

$$\sum_k \to 2\left(\frac{L}{2\pi}\right)^3 \int d^3k$$

spin degeneracy

Case of Fermi spin ½ particles for *T* →*0*.

$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} \approx \begin{cases} 1 & \text{for } \varepsilon_k < \mu \\ 0 & \text{for } \varepsilon_k > \mu \end{cases}$$

$$N = \sum_k \langle n_k \rangle \to 2\left(\frac{L}{2\pi}\right)^3 \int_{\varepsilon_k < \mu} d^3k = 2\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} k_F^3$$

$$\mu = \frac{\hbar^2 k_F^2}{2m} \equiv \varepsilon_F$$

$$N = \frac{V}{3\pi^2}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2}$$

$$\Rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

Case of Fermi spin ½ particles for $T \to 0$ continued.

$$N = \sum_k \langle n_k \rangle \to 2\left(\frac{L}{2\pi}\right)^3 \int_{\varepsilon_k<\mu} d^3k = 2\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{3} k_F^3$$

$$\langle E \rangle = \sum_k \langle n_k \rangle \varepsilon_k \to 2\left(\frac{L}{2\pi}\right)^3 \int_{\varepsilon_k<\mu} d^3k \frac{\hbar^2 k^2}{2m} = 2\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{5} \frac{\hbar^2 k_F^5}{2m}$$

$$\frac{\langle E \rangle}{N} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}$$

$$\frac{\langle E \rangle}{N} = \frac{3}{5} \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

$$P = -\left(\frac{\partial \langle E \rangle}{\partial V}\right) = \frac{2}{5} \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{2/3} \frac{N}{V}$$

Case of Fermi spin ½ particles for low temperature

Heat capacity at low temperatures:

$$C = \frac{\partial \langle E \rangle}{\partial T} = \sum_k \frac{\partial \langle n_k \rangle}{\partial T} \varepsilon_k \to 2\left(\frac{L}{2\pi}\right)^3 \int d^3k \frac{\partial}{\partial T}\left(\frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}\right)\varepsilon_k$$

Note that: $2\left(\frac{L}{2\pi}\right)^3 \int d^3k \equiv \int d\varepsilon \, g(\varepsilon)$

where $g(\varepsilon) \equiv 2\left(\frac{L}{2\pi}\right)^3 \int d^3k \, \delta(\varepsilon - \varepsilon_k) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$

$$C = \frac{\partial \langle E \rangle}{\partial T} = \int d\varepsilon \, g(\varepsilon) \frac{\partial}{\partial T}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right)\varepsilon$$

Some useful integrals:

$$\int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x+1)^2} = \frac{\pi^2}{3}$$

$$\int_{-\infty}^{\infty} dx \frac{x e^x}{(e^x+1)^2} = 0$$

Let $x = \beta(\varepsilon - \mu) = \frac{\varepsilon - \mu}{kT}$

$$\frac{dx}{dT} = -\frac{\varepsilon - \mu}{kT^2} = -\frac{x}{T} \qquad \varepsilon = kTx + \mu$$

$$C = \frac{\partial \langle E \rangle}{\partial T} = \int_0^{\infty} d\varepsilon \, g(\varepsilon) \frac{\partial}{\partial T}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right)\varepsilon$$

$$= kT \int_{-\beta\mu}^{\infty} dx \, g(\varepsilon) \left(\frac{e^x}{(e^x+1)^2} \frac{x}{T}\right)(kTx + \mu)$$

$$\approx kT \int_{-\infty}^{\infty} dx \, g(\mu) \left(\frac{e^x}{(e^x+1)^2} \frac{x}{T}\right)(kTx + \mu) = \frac{\pi^2}{3} g(\mu) k^2 T$$

Case of Bose particles

Non-interacting spin 0 particles of mass *m* at low *T* moving in 3-dimensions in large box of volume $V=L^3$:
Assume that each state e_k is singly occupied.

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2\left(n_x^2 + n_y^2 + n_z^2\right)}{8mL^2} \quad n_x, n_y, n_z = 1,2,3\cdots$$

$$\text{In the limit } L \to \infty,\ \varepsilon_k \to \frac{\hbar^2 k^2}{2m}$$

$$\sum_k \to \left(\frac{L}{2\pi}\right)^3 \int d^3k = \int d\varepsilon\, g_B(\varepsilon)$$

$$g_B(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

Case of Bose particles at low *T*

$$N = \sum_k \langle n_k \rangle = \langle n_0 \rangle + \sum_{k>0} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$N = \langle n_0 \rangle + \int_0^\infty d\varepsilon\, g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

Note that for low T a consistent solutions exists such that $N \approx \langle n_0 \rangle$

$$\langle n_0 \rangle = \frac{1}{e^{\beta(-\mu)} - 1} \approx \frac{1}{1 - \beta\mu - 1} = \frac{1}{-\beta\mu} \quad \text{assuming } |\mu| \text{ small}$$

$$\text{In this case, } \mu \approx -\frac{kT}{N}$$

Critical temperature for Bose condensation:

$$N = \underbrace{\langle n_0 \rangle}_{\text{condensate}} + \underbrace{\int_0^\infty d\varepsilon\, g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}}_{\text{“normal” state}}$$

If $N = \int_0^\infty d\varepsilon\, g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$, there is no "condensate"

The temperature at which the above equality is satisfied is called the Einstein condensation temperature T_E.

Approximate value:

$$N \approx \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \sqrt{\varepsilon} \frac{1}{e^{\beta_E \varepsilon} - 1} = \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} \int_0^\infty dx \sqrt{x} \frac{1}{e^x - 1}$$

$$N \approx \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} 2.612 \frac{\sqrt{\pi}}{2} \quad \Rightarrow kT_E = \left(\frac{N/V}{2.612}\right)^{2/3} \frac{2\pi\hbar^2}{m}$$