PHY 341/641
Thermodynamics and Statistical Physics

Lecture 24

Many particle systems (Chapter 6 in STP)

- Non-interacting Bose particles
  - Bose condensation
  - Photons
  - Phonons

17	2/24/2012	Introduction to magnetic systems	5.1-5.5		
	2/27/2012	APS -- no class; take-home exam			
	2/29/2012	APS -- no class; take-home exam			
	3/02/2012	APS -- no class; take-home exam			
18	3/05/2012	Exam due -- Ising model	5.5	HW 17	03/07/2012
19	3/07/2012	Ising model	5.6-5.7	HW 18	03/09/2012
20	3/09/2012	Phase transformation	5.8-5.10		
	3/12/2012	*Spring Break*			
	3/14/2012	*Spring Break*			
	3/16/2012	*Spring Break*			
21	3/19/2012	Many particle systems	6.1-6.2	HW 19	03/23/2012
22	3/21/2012	Fermi and Bose particles	6.3-6.4		
23	3/23/2012	Bose and Fermi particles	6.5-6.11	HW 20	03/28/2012
24	3/26/2012	Bose and Fermi particles	6.5-6.11		
25	3/28/2012	Chemical potential	7.1-7.2		
26	3/30/2012	Phase equilibria	7.3		
	4/02/2012				

Reminder – second exam in April

-- student presentations 4/30, 5/2 (need to pick topics)

Case of Bose particles

Non-interacting spin 0 particles of mass *m* at low *T* moving in 3-dimensions in large box of volume $V=L^3$:

Assume that each state e_k is singly occupied.

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2\left(n_x^2 + n_y^2 + n_z^2\right)}{8mL^2} \quad n_x, n_y, n_z = 1,2,3\cdots$$

$$\text{In the limit } L \to \infty,\ \varepsilon_\text{k} \to \frac{\hbar^2 k^2}{2m}$$

$$\sum_k \;\to \left(\frac{L}{2\pi}\right)^3 \int d^3k = \int d\varepsilon\; g_B(\varepsilon)$$

$$g_B(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

Case of Bose particles at low *T*

$$N = \sum_k \langle n_k \rangle = \langle n_0 \rangle + \sum_{k>0} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$N = \langle n_0 \rangle + \int_0^\infty d\varepsilon \; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

Note that for low *T* a consistent solutions exists such that

$$N \approx \langle n_0 \rangle$$

$$\langle n_0 \rangle = \frac{1}{e^{\beta(-\mu)} - 1} \approx \frac{1}{1 - \beta\mu - 1} = \frac{1}{-\beta\mu} \quad \text{assuming } |\mu| \text{ small}$$

In this case, $\mu \approx -\frac{kT}{N}$

Critical temperature for Bose condensation:

$$N = \langle n_0 \rangle + \int_0^\infty d\varepsilon\; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

$$\underbrace{\langle n_0 \rangle}_{\text{condensate}} \qquad \underbrace{\int_0^\infty d\varepsilon\; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1}}_{\text{"normal" state}}$$

If $N = \int_0^\infty d\varepsilon\; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$, there is no "condensate"

The temperature at which the above equality is satisfied is called the Einstein condensation temperature T_E.

Approximate value:

$$N \approx \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \sqrt{\varepsilon}\, \frac{1}{e^{\beta_E \varepsilon}-1} = \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} \int_0^\infty dx \sqrt{x}\, \frac{1}{e^{x}-1}$$

$$N \approx \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} 2.612 \frac{\sqrt{\pi}}{2} \quad \Rightarrow kT_E = \left(\frac{N/V}{2.612}\right)^{2/3} \frac{2\pi\hbar^2}{m}$$

Systematic treatment of Bose case
(Ref. L. E. Reichl, **A Modern Course in Statistical Mechanics**, Wiley-Interscience 1998, J. D. Walecka, **Introduction to Statistical Mechanics**, World Scientific 2011)

$$\text{Define} \quad z \equiv e^{\beta\mu} \leq 1 \qquad \lambda_T \equiv \left(\frac{2\pi\hbar^2}{mkT} \right)^{1/2}$$

The Landau potential for the Bose system can be written :

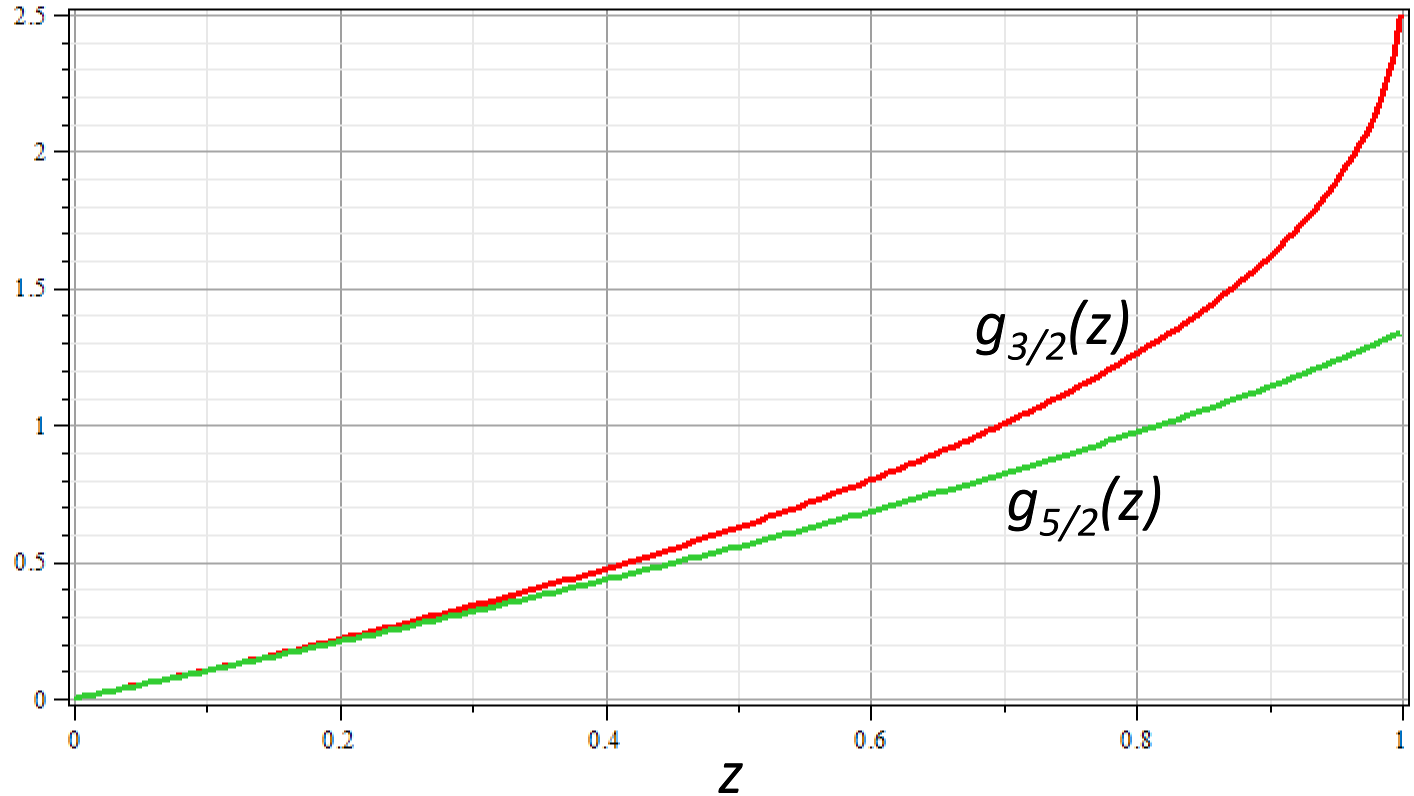
$$\Omega_B(T, V, \mu) = kT \ln(1-z) + \frac{4kT}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T\sqrt{\pi}/L}^{\infty} dx\, x^2 \ln\left(1 - z e^{-x^2}\right)$$

$$\langle N \rangle = -\left(\frac{\partial \Omega_B}{\partial \mu} \right) = \underbrace{\frac{z}{1-z}}_{\langle n_0 \rangle} + \frac{4}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T\sqrt{\pi}/L}^{\infty} dx\, x^2 \left(\frac{z}{e^{x^2} - z} \right)$$

Some convenient integrals

$$g_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \ln\left(1 - z e^{-x^2}\right) = \sum_{n=1}^\infty \frac{z^n}{n^{5/2}}$$

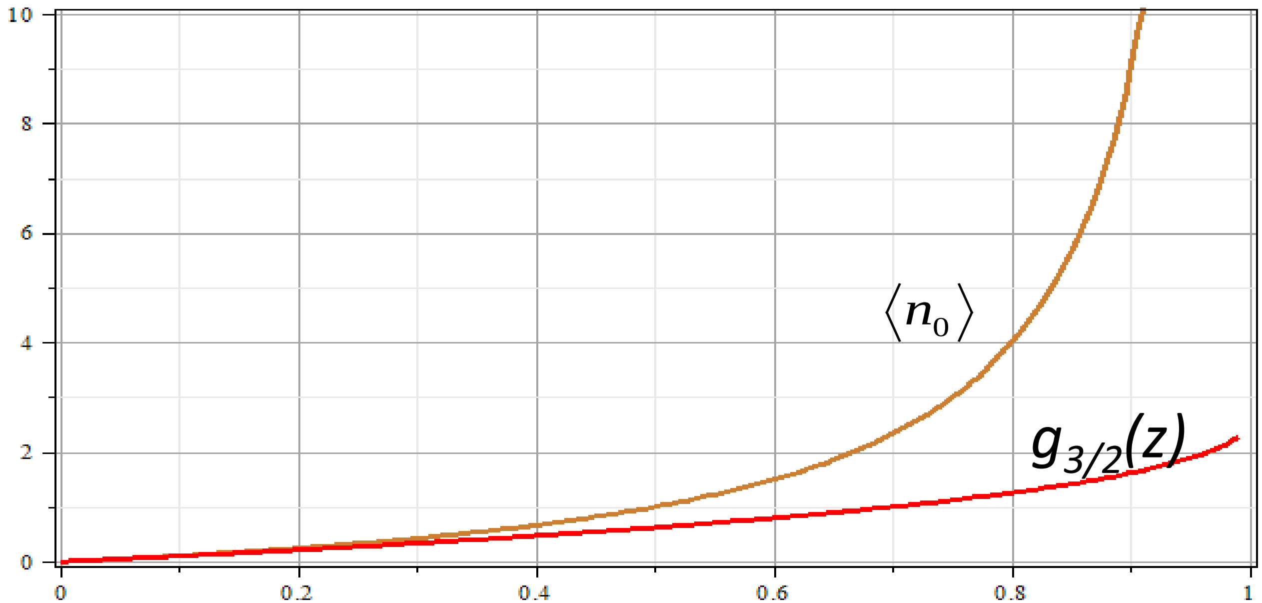
$$g_{3/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right) = z \frac{d}{dz} g_{5/2}(z) = \sum_{n=1}^\infty \frac{z^n}{n^{3/2}}$$



$$\langle n_0 \rangle = \frac{z}{1-z} \qquad \text{where } z \equiv e^{\beta\mu} \le 1 \qquad \lambda_T \equiv \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$

$$\langle N \rangle - \langle n_0 \rangle = \frac{4}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T \sqrt{\pi}/L}^{\infty} dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right) = \frac{V}{\lambda_T^3} \left(g_{3/2}(z) - \frac{4}{\sqrt{\pi}} \int_0^{\lambda_T \sqrt{\pi}/L} dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right) \right)$$

$$\langle N \rangle - \langle n_0 \rangle \approx \frac{V}{\lambda_T^3} g_{3/2}(z) \qquad \text{note that} \quad \lim_{z \to 1} g_{3/2}(z) = 2.612$$



z

Equation for z:

$$\langle N \rangle = \langle n_0 \rangle + \frac{V}{\lambda_T^3} g_{3/2}(z)$$

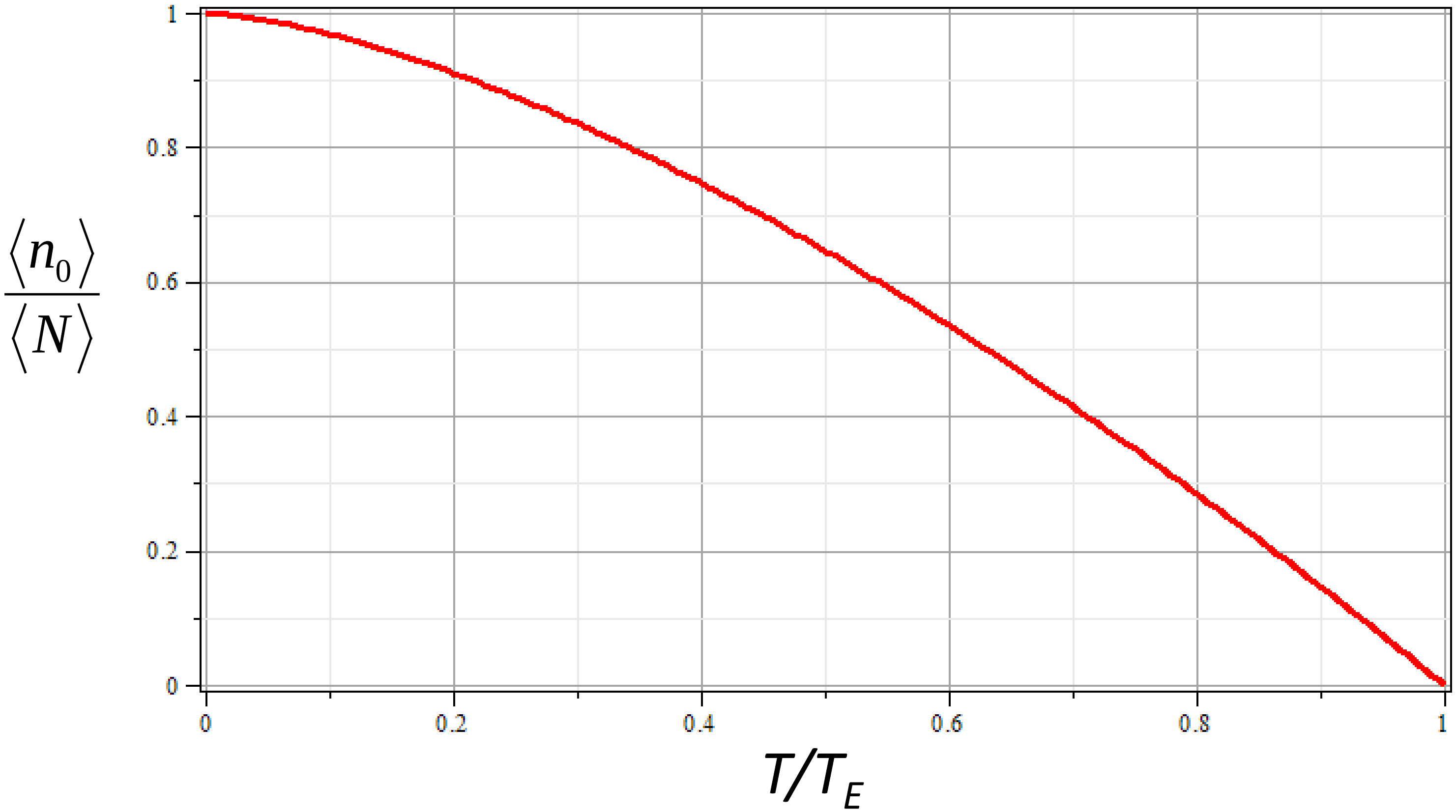
Define Einstein temperature

$$\langle N \rangle = \frac{V}{\lambda_{T_E}^3} g_{3/2}(1) = \frac{V}{\lambda_{T_E}^3} 2.612$$

For $T > T_E$, $\quad \langle n_0 \rangle << \langle N \rangle$ and $\langle N \rangle = \frac{V}{\lambda_T^3} g_{3/2}(z)$ has a solution for $z < 1$

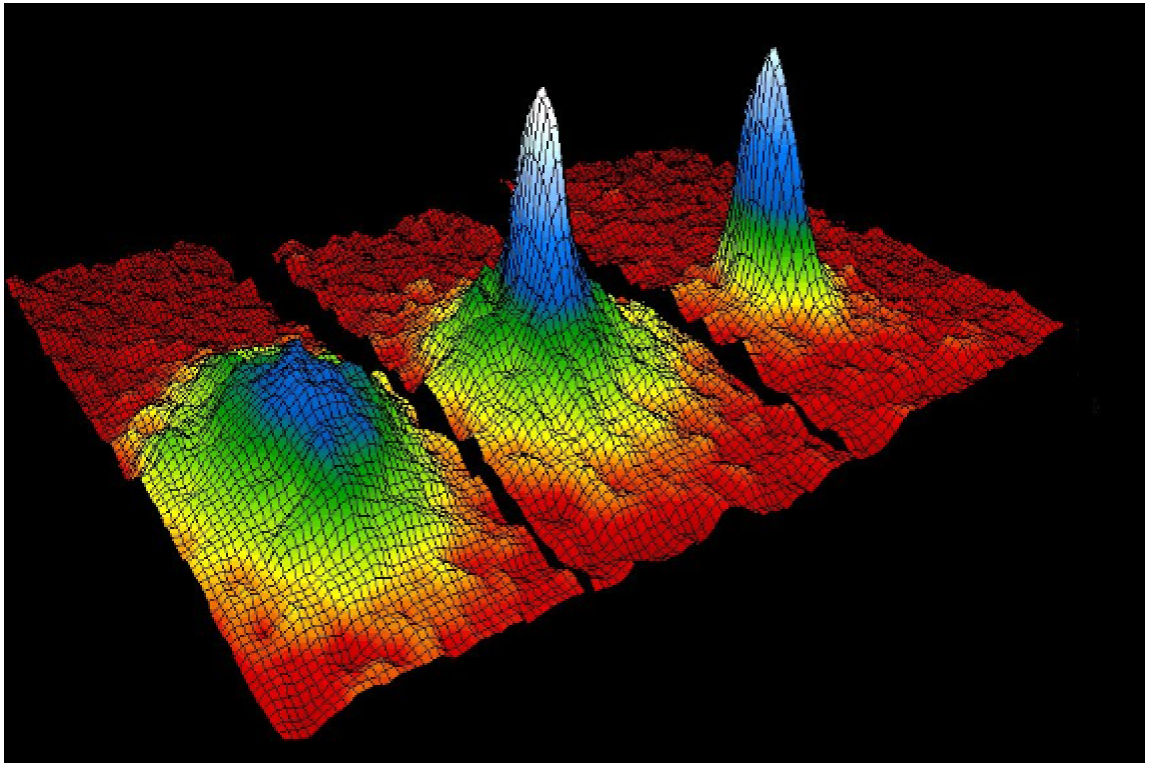
For $T \leq T_E$, $\quad z \to 1^-$ and $\langle N \rangle = \langle n_0 \rangle + \frac{V}{\lambda_T^3} g_{3/2}(1)$

$\Rightarrow$ For $T \leq T_E$ $\quad \frac{\langle n_0 \rangle}{\langle N \rangle} = 1 - \frac{\lambda_{T_E}^3}{\lambda_T^3} = 1 - \left(\frac{T}{T_E} \right)^{3/2}$



$\frac{\langle n_0 \rangle}{\langle N \rangle}$
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
T/T_E

http://www.colorado.edu/physics/2000/bec/three_peaks.html



Bose-Einstein Condensation at 400, 200, and 50 nano-Kelvins

^{87}Rb atoms (~2000 atoms in condensate)

The Nobel Prize in Physics 2001
Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman

The Nobel Prize in Physics 2001

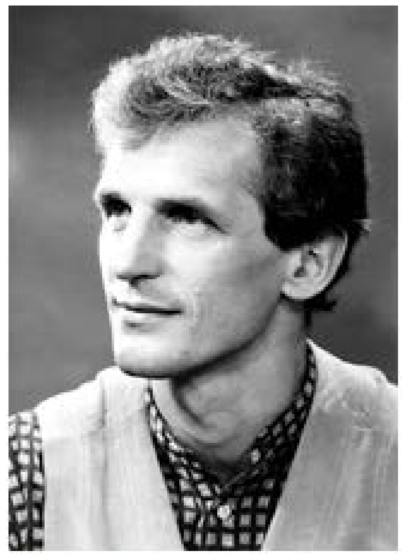
Nobel Prize Award Ceremony

Eric A. Cornell

Wolfgang Ketterle

Carl E. Wieman


Eric A. Cornell


Wolfgang Ketterle


Carl E. Wieman

The Nobel Prize in Physics 2001 was awarded jointly to Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman *"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"*.

Other systems with Bose statistics

Thermal distribution of photons -- blackbody radiation:

In this case, the number of particles (photons) is not conserved so that μ=0.

$$\langle n_k \rangle = \frac{1}{e^{\beta\varepsilon_k} - 1}$$

$$\varepsilon_k = \hbar\omega = \hbar ck = h\nu$$

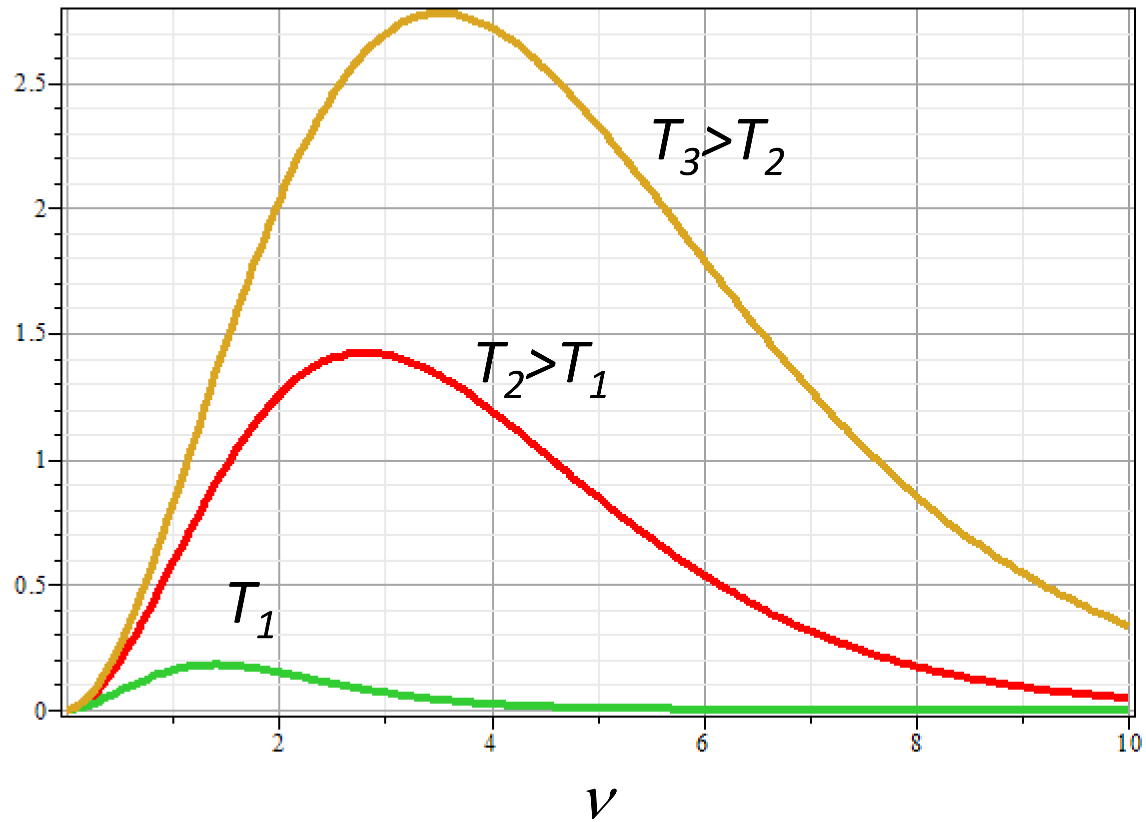
$$\sum_k \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d\varepsilon \int d^3k\,\delta(\varepsilon - \hbar ck) = \frac{V}{\pi^2\hbar^3c^3}\int d\varepsilon\ \varepsilon^2$$

Distribution of radiated energy:

$$\langle E \rangle = \sum_k \langle n_k \rangle \varepsilon_k = \frac{V}{\pi^2\hbar^3c^3}\int d\varepsilon\ \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1} = \frac{8\pi hV}{c^3}\int d\nu\, \frac{\nu^3}{e^{\beta h\nu} - 1}$$

$$\langle E \rangle = \frac{8\pi^5 V (kT)^4}{15(hc)^3}$$

Blackbody radiation distribution:



Other systems with Bose statistics

Thermal distribution of vibrations -- phonons:

In this case, the number of particles (phonons) is not conserved so that μ=0.

$$\langle n_k \rangle = \frac{1}{e^{\beta\varepsilon_k} - 1}$$

$$\varepsilon_k = \hbar\omega$$

For Einstein solid, the fundamental frequency ω vibrates in 3 directions for all N particles.

$$\langle E \rangle = 3N\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right)$$

$$C = \left(\frac{\partial\langle E \rangle}{\partial T}\right) = 3Nk\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2}$$

Other systems with Bose statistics -- continued

Thermal distribution of vibrations -- phonons:

$$\langle n_k \rangle = \frac{1}{e^{\beta \varepsilon_k} - 1}$$

$$\varepsilon_k = \hbar \omega$$

For Debye solid, the fundamental frequency $\omega = \bar{c}k$, where $\bar{c}$ denotes the speed of sound (assumed here to be the same in 3 directions).

$$\langle E \rangle = \frac{3V\hbar}{2\pi^2 \bar{c}^3} \int_0^{\omega_D} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1} = 9NkT \left(\frac{T}{T_D} \right)^3 \int_0^{T_D/T} \frac{x^3 dx}{e^x - 1}$$