PHY 341/641
Thermodynamics and Statistical Physics

Lecture 24

Many particle systems (Chapter 6 in STP)

- Non-interacting Bose particles
  - Bose condensation
  - Photons
  - Phonons

17	2/24/2012	Introduction to magnetic systems	5.1-5.5		
	2/27/2012	APS -- no class; take-home exam			
	2/29/2012	APS -- no class; take-home exam			
	3/02/2012	APS -- no class; take-home exam			
18	3/05/2012	Exam due -- Ising model	5.5	HW 17	03/07/2012
19	3/07/2012	Ising model	5.6-5.7	HW 18	03/09/2012
20	3/09/2012	Phase transformation	5.8-5.10		
	3/12/2012	*Spring Break*			
	3/14/2012	*Spring Break*			
	3/16/2012	*Spring Break*			
21	3/19/2012	Many particle systems	6.1-6.2	HW 19	03/23/2012
22	3/21/2012	Fermi and Bose particles	6.3-6.4		
23	3/23/2012	Bose and Fermi particles	6.5-6.11	HW 20	03/28/2012
24	3/26/2012	Bose and Fermi particles	6.5-6.11		
25	3/28/2012	Chemical potential	7.1-7.2		
26	3/30/2012	Phase equilibria	7.3		
	4/02/2012				

Reminder – second exam in April
-- student presentations 4/30, 5/2 (need to pick topics)

Case of Bose particles

Non-interacting spin 0 particles of mass *m* at low *T* moving in 3-dimensions in large box of volume $V=L^3$:
Assume that each state e_k is singly occupied.

$$N = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2\left(n_x^2 + n_y^2 + n_z^2\right)}{8mL^2} \quad n_x, n_y, n_z = 1,2,3\cdots$$

$$\text{In the limit } L \to \infty, \ \varepsilon_k \to \frac{\hbar^2 k^2}{2m}$$

$$\sum_k \to \left(\frac{L}{2\pi}\right)^3 \int d^3k = \int d\varepsilon \, g_B(\varepsilon)$$

$$g_B(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

Case of Bose particles at low *T*

$$N = \sum_k \langle n_k \rangle = \langle n_0 \rangle + \sum_{k>0} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$N = \langle n_0 \rangle + \int_0^\infty d\varepsilon \; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

Note that for low T a consistent solutions exists such that $N \approx \langle n_0 \rangle$

$$\langle n_0 \rangle = \frac{1}{e^{\beta(-\mu)} - 1} \approx \frac{1}{1 - \beta\mu - 1} = \frac{1}{-\beta\mu} \text{ assuming } |\mu| \text{ small}$$

In this case, $\mu \approx -\frac{kT}{N}$

3/26/2012 PHY 341/641 Spring 2012 -- Lecture 24 4

Critical temperature for Bose condensation:

$$\underbrace{N = \langle n_0 \rangle}_{\text{condensate}} + \underbrace{\int_0^\infty d\varepsilon \; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}}_{\text{“normal” state}}$$

condensate “normal” state

If $N = \int_0^\infty d\varepsilon \; g_B(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$, there is no "condensate"

The temperature at which the above equality is satisfied is called the Einstein condensation temperature T_E.

Approximate value:

$$N \approx \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \sqrt{\varepsilon} \frac{1}{e^{\beta_E \varepsilon} - 1} = \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} \int_0^\infty dx \sqrt{x} \frac{1}{e^x - 1}$$

$$N \approx \frac{V}{4\pi^2}\left(\frac{2mkT_E}{\hbar^2}\right)^{3/2} 2.612 \frac{\sqrt{\pi}}{2} \quad \Rightarrow kT_E = \left(\frac{N/V}{2.612}\right)^{2/3} \frac{2\pi\hbar^2}{m}$$

3/26/2012 PHY 341/641 Spring 2012 -- Lecture 24 5

Systematic treatment of Bose case
(Ref. L. E. Reichl, **A Modern Course in Statistical Mechanics**, Wiley-Interscience 1998, J. D. Walecka, **Introduction to Statistical Mechanics**, World Scientific 2011)

Define $z \equiv e^{\beta\mu} \le 1 \qquad \lambda_T \equiv \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$

The Landau potential for the Bose system can be written:

$$\Omega_B(T, V, \mu) = kT \ln(1 - z) + \frac{4kT}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T\sqrt{\pi}/L}^\infty dx \, x^2 \ln\left(1 - ze^{-x^2}\right)$$

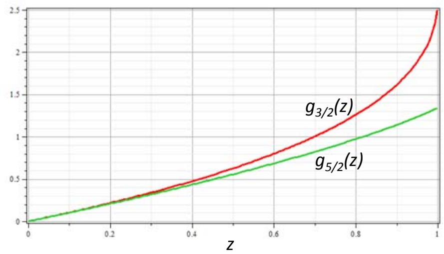
$$\langle N \rangle = -\left(\frac{\partial \Omega_B}{\partial \mu}\right) = \underbrace{\frac{z}{1 - z}}_{\langle n_0 \rangle} + \frac{4}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T\sqrt{\pi}/L}^\infty dx \, x^2 \left(\frac{z}{e^{x^2} - z}\right)$$

3/26/2012 PHY 341/641 Spring 2012 -- Lecture 24 6

Some convenient integrals

$$g_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \ln\left(1 - ze^{-x^2}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}}$$

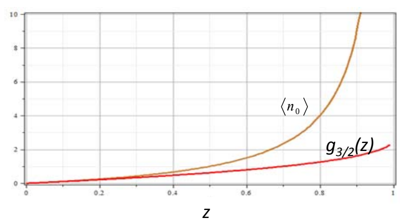
$$g_{3/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right) = z \frac{d}{dz} g_{5/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}}$$



$$\langle n_0 \rangle = \frac{z}{1-z} \qquad \text{where } z \equiv e^{\beta\mu} \le 1 \qquad \lambda_T \equiv \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$

$$\langle N \rangle - \langle n_0 \rangle = \frac{4}{\sqrt{\pi}} \frac{V}{\lambda_T^3} \int_{\lambda_T\sqrt{\pi}/L}^{\infty} dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right) = \frac{V}{\lambda_T^3}\left(g_{3/2}(z) - \frac{4}{\sqrt{\pi}} \int_0^{\lambda_T\sqrt{\pi}/L} dx\, x^2 \left(\frac{z}{e^{x^2} - z}\right)\right)$$

$$\langle N \rangle - \langle n_0 \rangle \approx \frac{V}{\lambda_T^3} g_{3/2}(z) \qquad \text{note that } \lim_{z\to 1} g_{3/2}(z) = 2.612$$



Equation for z:

$$\langle N \rangle = \langle n_0 \rangle + \frac{V}{\lambda_T^3} g_{3/2}(z)$$

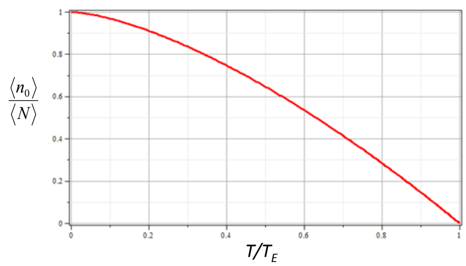
Define Einstein temperature

$$\langle N \rangle = \frac{V}{\lambda_{T_E}^3} g_{3/2}(1) = \frac{V}{\lambda_{T_E}^3} 2.612$$

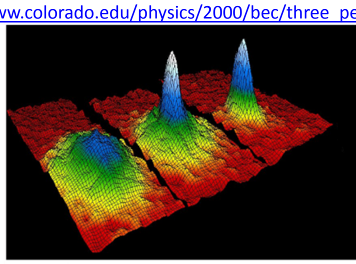
For $T > T_E$, $\langle n_0 \rangle << \langle N \rangle$ and $\langle N \rangle = \frac{V}{\lambda_T^3} g_{3/2}(z)$ has a solution for $z < 1$

For $T \le T_E$, $z \to 1^-$ and $\langle N \rangle = \langle n_0 \rangle + \frac{V}{\lambda_T^3} g_{3/2}(1)$

$$\Rightarrow \text{For } T \le T_E \quad \frac{\langle n_0 \rangle}{\langle N \rangle} = 1 - \frac{\lambda_{T_E}^3}{\lambda_T^3} = 1 - \left(\frac{T}{T_E}\right)^{3/2}$$

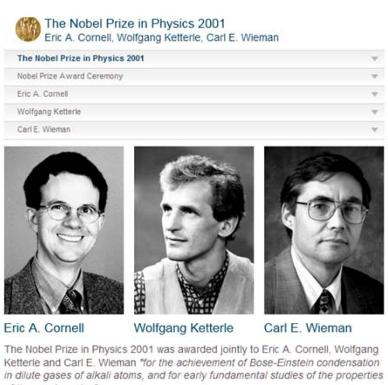


http://www.colorado.edu/physics/2000/bec/three_peaks.html



Bose-Einstein Condensation at 400, 200, and 50 nano-Kelvins

^{87}Rb atoms (~2000 atoms in condensate)



Other systems with Bose statistics

Thermal distribution of photons -- blackbody radiation:

In this case, the number of particles (photons) is not conserved so that μ=0.

$$\langle n_k \rangle = \frac{1}{e^{\beta\varepsilon_k}-1}$$

$$\varepsilon_k = \hbar\omega = \hbar ck = h\nu$$

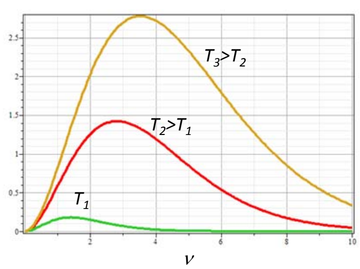
$$\sum_k \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d\varepsilon \int d^3k\,\delta(\varepsilon - \hbar ck) = \frac{V}{\pi^2\hbar^3c^3}\int d\varepsilon\,\varepsilon^2$$

Distribution of radiated energy:

$$\langle E \rangle = \sum_k \langle n_k \rangle \varepsilon_k = \frac{V}{\pi^2\hbar^3c^3}\int d\varepsilon \frac{\varepsilon^3}{e^{\beta\varepsilon}-1} = \frac{8\pi hV}{c^3}\int d\nu \frac{\nu^3}{e^{\beta h\nu}-1}$$

$$\langle E \rangle = \frac{8\pi^5 V (kT)^4}{15(hc)^3}$$

Blackbody radiation distribution:



Other systems with Bose statistics

Thermal distribution of vibrations -- phonons:

In this case, the number of particles (phonons) is not conserved so that μ=0.

$$\langle n_k \rangle = \frac{1}{e^{\beta\varepsilon_k}-1}$$

$$\varepsilon_k = \hbar\omega$$

For Einstein solid, the fundamental frequency ω vibrates in 3 directions for all N particles.

$$\langle E \rangle = 3N\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}+\frac{1}{2}\right)$$

$$C = \left(\frac{\partial\langle E \rangle}{\partial T}\right) = 3Nk\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

Other systems with Bose statistics -- continued

Thermal distribution of vibrations -- phonons:

$$\langle n_k \rangle = \frac{1}{e^{\beta \varepsilon_k} - 1}$$

$$\varepsilon_k = \hbar \omega$$

For Debye solid, the fundamental frequency $\omega = \bar{c}k$, where $\bar{c}$ denotes the speed of sound (assumed here to be the same in 3 directions).

$$\langle E \rangle = \frac{3V\hbar}{2\pi^2 \bar{c}^3} \int_0^{\omega_D} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1} = 9NkT\left(\frac{T}{T_D}\right)^3 \int_0^{T_D/T} \frac{x^3 dx}{e^x - 1}$$

3/26/2012 PHY 341/641 Spring 2012 -- Lecture 24 16