PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 14:

Start reading Chapter 6

1. **Maxwell's full equations; effects of time varying fields and sources**
2. **Gauge choices and transformations**
3. **Green's function for vector and scalar potentials**

	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/05/2014
8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/19/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/19/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/21/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Dynamic dielectric media and their effects		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
	Mon: 03/17/2014		Take-home exam due		

Full electrodynamics with time varying fields and sources

Maxwell's equations

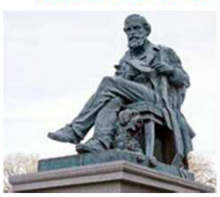

Image of statue of James Clerk-Maxwell in Edinburgh

"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"

Richard P Feynman

http://www.clerkmaxwellfoundation.org/

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P} = 0;\ \mathbf{M} = 0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = \mu_0 \mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = \mu_0 \mathbf{J}$$

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2}\frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_l = 0 \qquad \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_l = -\varepsilon_0 \nabla \cdot \frac{\partial(\nabla \Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial(\nabla \Phi_C)}{\partial t} = \varepsilon_0 \mu_0 \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}_l$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Analysis of the scalar and vector potential equations :

$$-\nabla^2\Phi - \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Alternate potentials : $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial\Lambda}{\partial t}$

Yields same physics provided that : $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$

Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2\Phi_L - \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = -\rho/\varepsilon_0$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0\mathbf{J}$$

Consider the general form of the 3-dimensional wave equation :

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r},t) \Rightarrow$ wave field $\quad f(\mathbf{r},t) \Rightarrow$ source

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\nabla^2\Psi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2}=-4\pi f(\mathbf{r},t)$$

Green's function :

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+\int d^3r'\int dt' G(\mathbf{r},t;\mathbf{r}',t')f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function :

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity :

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+$$
$$\int d^3r'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function :

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Fourier analysis in the time domain -- note that

$$\delta(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}$$

Define :

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}\,\widetilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$\Rightarrow\quad\left(\nabla^2+\frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega)=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$$

Further assuming that $\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$ is isotropic in $|\mathbf{r}-\mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution: $\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{R}e^{\pm i\omega R/c}$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\,\widetilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c)}\right)$$

$$= \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t\mp|\mathbf{r}-\mathbf{r}'|/c)$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t\pm|\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) + \int d^3r'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liènard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge *q* moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\,\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t)),$ where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}.$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt'\frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R-\frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$ $\quad t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right) - \frac{\mathrm{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot R}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right] = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

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