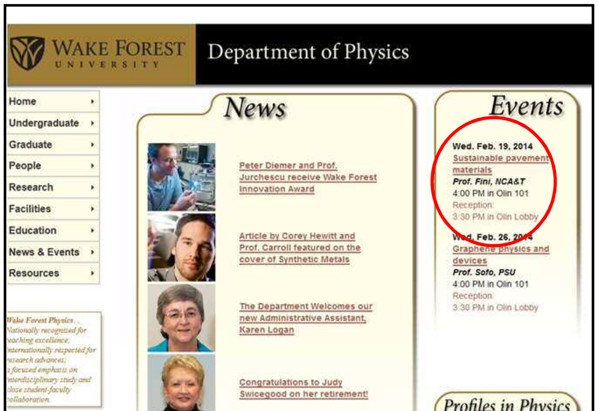
PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 15:

Finish reading Chapter 6

1. Some details of Liénard-Wiechert results
2. Energy density and flux associated with electromagnetic fields
3. Time harmonic fields

	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/05/2014
8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/19/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/19/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/21/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Dynamic dielectric media and their effects		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
	Mon: 03/17/2014		Take-home exam due		



WFU Physics Colloquium

TITLE: Production and Characterization of Bio-based Adhesives for Construction Applications

SPEAKER: Professor Elham (Ellie) H. Fini,

Department of Civil Engineering,
North Carolina A&T University

TIME: Wednesday February 19, 2014 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

To enhance economic, environmental and social well-being both private and public agencies are emphasizing the need for adopting more "sustainable" practices and products in design, construction, and maintenance of infrastructure, including pavements. The trend toward sustainable pavements has led the pavement industry to place more emphasis on application of new materials such as warm mix asphalt (WMA), half-warm mix asphalt (HWMA) and cold mix asphalt (CMA) in order to reduce carbon footprints of pavement and to reduce fuel consumption and CO2 emission. Depleting aggregate resources and a stricter regulatory environment also led exploitation schemes to evolve towards greater recycling emphasizing on increasing percentages of reclaimed asphalt pavements (RAP).

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}']|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r]_{d^3r'}dt'\frac{\mathbf{J}(\mathbf{r}',\mathrm{t}')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\right).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Note that for any function *F(x)*:

$$\int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx = F(x_0)$$

Now consider a function *p(x)*, for which $p(x_i) = 0$ for $i = 1, 2, \cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left((x - x_i)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

In this case we have: $\int_{-\infty}^{\infty} f(t')\delta\left(p(t')\right)dt' = \frac{f(t_r)}{\left|1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t_r)\right|}\right|}$

where: $p(t') \equiv t' - \left(t - \frac{\left|\mathbf{r}-\mathbf{R}_q(t')\right|}{c}\right)$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'}\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t')\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|}$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r), \qquad t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1-\frac{v^2}{c^2}\right) - \frac{\mathbf{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot R}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1-\frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1-\frac{v^2}{c^2}+\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right] = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \; \mathbf{E}\cdot\mathbf{J}$$

Expressing source current in terms of fields it produces:

$$\frac{dW}{dt} = \int d^3r \; \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dW}{dt} = \int d^3r\ \mathbf{E}\cdot\mathbf{J} = \int d^3r\ \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$= -\int d^3r\ \left(\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r\ \mathbf{E}\cdot\mathbf{J}$$

Electromagnetic energy density: $u \equiv \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$

$$E_{field} \equiv \int d^3r\ u(\mathbf{r},t)$$

Poynting vector: $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$

From the previous energy analysis: $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r\ \nabla\cdot\mathbf{S}(\mathbf{r},t) = -\oint d^2r\ \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r\ (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})$$

Follows by analogy with Lorentz force:

$$F = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r\ (\mathbf{E}\times\mathbf{B})$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r\ \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\right)$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \widetilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow$ $\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \widetilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Coulomb's law:	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \widetilde{\mathbf{D}} = \widetilde{\rho}_{free}$
Ampere - Maxwell's law:	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \widetilde{\mathbf{H}} + i\omega\widetilde{\mathbf{D}} = \widetilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \widetilde{\mathbf{E}} - i\omega\widetilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \widetilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right) \times \left(\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega) + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\langle \mathbf{S}(\mathbf{r},t) \rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

1. Suppose that an electromagnetic wave of pure (real) frequency ω is traveling along the z-axis of a wave guide having a square cross section with side dimension a composed of a medium having a real permittivity constant ε and a real permeability constant μ. Suppose that the wave is known to have the form:

$$\mathbf{E}(\mathbf{r},t) = \Re\left\{H_0 e^{ikz-i\omega t}\left[(i\mu\omega)\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{y}}\right]\right\}$$

$$\mathbf{H}(\mathbf{r},t) = \Re\left\{H_0\ e^{ikz-i\omega t}\left[-ik\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\ \hat{\mathbf{x}} + \cos\left(\frac{\pi x}{a}\right)\ \hat{\mathbf{z}}\right]\right\}.$$

Here H_0 denotes a real amplitude, and the parameter k is assumed to be real and equal to

$$k \equiv \sqrt{\mu\epsilon\omega^2 - \left(\frac{\pi}{a}\right)^2}, \quad \text{for} \quad \mu\epsilon\omega^2 > \left(\frac{\pi}{a}\right)^2$$

a. Show that this wave satisfies the sourceless Maxwell's equations.
b. Find the form of the time-averaged Poynting vector

$$\langle \mathbf{S}\rangle_{avg} \equiv \frac{1}{2}\Re\left\{\mathbf{E}(\mathbf{r},t)\times\mathbf{H}^*(\mathbf{r},t)\right\}$$

for this electromagnetic wave.

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Summary and review

Maxwell's equations

Coulomb's law: $\nabla\cdot\mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

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Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon\mathbf{E}$; $\mathbf{B} = \mu\mathbf{H}$

and no sources:

Coulomb's law: $\nabla\cdot\mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B} - \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

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Analysis of Maxwell's equations without sources -- continued:

Coulomb's law : $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law : $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

where $v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation :

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$$

Note: *ε, μ, n, k* can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

from Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

also note : $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$ and $\hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle \mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \times \frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$