PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 19:

Start reading Chap. 8 in Jackson.

A. Examples of waveguides

B. TEM, TE, TM modes

C. Resonant cavities

	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/05/2014
8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/28/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/28/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/28/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Complex dielectrics; TEM modes		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
19	Mon: 03/17/2014	Chap. 8	Wave guides;Take-home exam due	#18	3/19/2014
20	Wed: 03/19/2014	Chap. 8	Wave guides	#19	3/21/2014

Reminder:
- Topic choices for "computational project"
– suggestions available upon request
-- due Monday March 24th?

Fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2} \right) \mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r}, t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \right) \qquad \text{where } \mathbf{k} = (n_R + i n_I) \frac{\omega}{c} \hat{\mathbf{k}}$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c} n_R = \omega \sqrt{\frac{\mu\varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b} \right)^2} + 1 \right)^{1/2}$$

$$\frac{\omega}{c} n_I = \omega \sqrt{\frac{\mu\varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b} \right)^2} - 1 \right)^{1/2}$$

For $\frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

$$\Rightarrow \mathbf{E}(\mathbf{r}, t) \approx e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t} \right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r}, t) \approx \Re\left(\frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \right) \approx \Re\left(\frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \right)$$

Fields near the surface on an ideal conductor -- continued

For $\frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

In this limit, $\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + i n_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$

$$\mathbf{E}(\mathbf{r}, t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t} \right)$$

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon \mathbf{E}(\mathbf{r}, t) = \frac{i\sigma}{\omega} \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\mu\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{B}(\mathbf{r}, t) = \mu \mathbf{H}(\mathbf{r}, t) = \frac{n}{c} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) = \frac{1+i}{\delta\omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$$

Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega}\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Note that we can express the results in terms of the $\mathbf{H}$ field:

$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\,\Re\left(\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)\right)$$

Boundary values for ideal conductor

Inside the conductor:

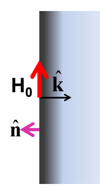
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

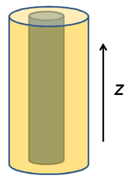
Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\Big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\Big|_S = 0$$

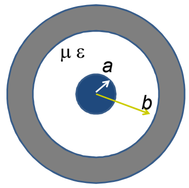


Waveguide terminology
- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

Wave guides



Top view:



Inside medium, μ ε assumed to be real

Coaxial cable
TEM modes

(following problem 8.2 in Jackson's text)

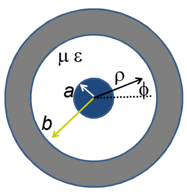
Maxwell's equations inside medium: for $a \leq \rho \leq b$

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = -i\omega\mu\varepsilon\mathbf{E} \qquad \nabla \cdot \mathbf{B} = 0$$

Electromagnetic waves in a coaxial cable -- continued

Top view:



Example solution for $a \leq \rho \leq b$

$$\mathbf{E} = \hat{\boldsymbol{\rho}}\Re\left(\frac{E_0 a}{\rho} e^{ikz-i\omega t}\right)$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\Re\left(\frac{B_0 a}{\rho} e^{ikz-i\omega t}\right)$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Find:

$$k = \omega\sqrt{\mu\varepsilon}$$

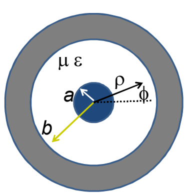
$$E_0 = \frac{B_0}{\sqrt{\mu\varepsilon}}$$

Poynting vector within cable medium (with μ, ε):

$$\langle \mathbf{S} \rangle_{avg} = \frac{1}{2\mu}\Re\left(\mathbf{E} \times \mathbf{B}^*\right) = \frac{|B_0|^2}{2\mu\sqrt{\mu\varepsilon}}\left(\frac{a}{\rho}\right)^2 \hat{\mathbf{z}}$$

Electromagnetic waves in a coaxial cable -- continued

Top view:

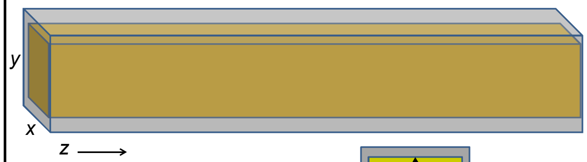


Time averaged power in cable material:

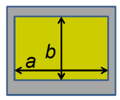
$$\int_0^{2\pi} d\phi \int_a^b \rho d\rho \left(\langle \mathbf{S} \rangle_{avg} \cdot \hat{\mathbf{z}}\right) = \frac{|B_0|^2 \pi a^2}{\mu\sqrt{\mu\varepsilon}} \ln\left(\frac{b}{a}\right)$$

Analysis of rectangular waveguide

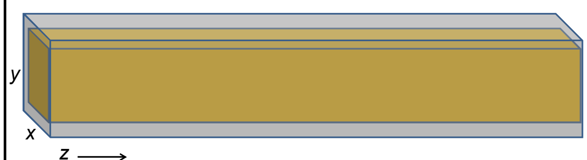
Boundary conditions at surface of waveguide:
$\mathbf{E}_{tangential}=0$, $\mathbf{B}_{normal}=0$



Cross section view



Analysis of rectangular waveguide



$$\mathbf{B} = \Re\left\{\left(B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}} + B_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$
$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$
$$k^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Maxwell's equations within the pipe:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0.$$
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$$
$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \qquad \frac{\partial B_z}{\partial y} - ikB_y = -i\mu\varepsilon\omega E_x.$$
$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y. \qquad ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\varepsilon\omega E_y.$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z. \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\varepsilon\omega E_z.$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)E_x(x,y) = 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

with $k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$

Some of the other field components are:

$$B_x = -\frac{k}{\omega}E_y \quad \text{and} \quad B_y = \frac{k}{\omega}E_x.$$

TE modes for retangular wave guide continued:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x=\frac{\omega}{k}B_y=\frac{-i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y}=\frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b}B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y=-\frac{\omega}{k}B_x=\frac{i\omega}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial x}=\frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a}B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Check boundary conditions:

$\mathbf{E}_{\text{tangential}}=0$ because: $E_x(x,0)=E_x(x,b)=0$

and $E_y(0,y) = E_y(a,y) = 0.$

$\mathbf{B}_{\text{normal}}=0$

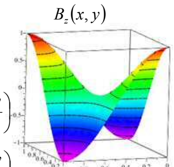
Solution for m=n=1

$$B_z(x,y)=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$iE_x(x,y)=B_0\left(\frac{\omega n\pi/b}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

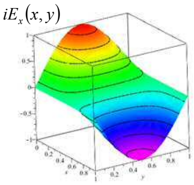
$$iE_y(x,y)=B_0\left(\frac{-\omega m\pi/a}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

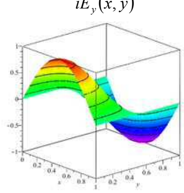
$B_z(x,y)$



$iE_y(x,y)$

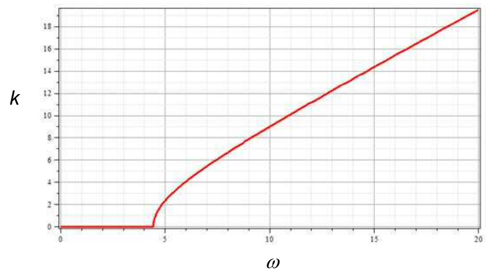
$iE_x(x,y)$



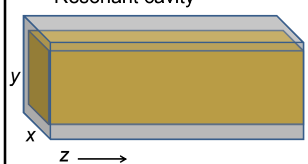


Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$



Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

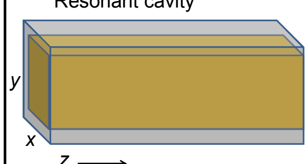
$$\mathbf{B} = \Re\left\{\left(B_x(x,y,z)\hat{\mathbf{x}} + B_y(x,y,z)\hat{\mathbf{y}} + B_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$
$$\mathbf{E} = \Re\left\{\left(E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}\right)e^{-i\omega t}\right\}$$

In general: $E_i(x,y,z) = E_i(x,y)\sin(kz)$ or $E_i(x,y)\cos(kz)$

$B_i(x,y,z) = B_i(x,y)\sin(kz)$ or $B_i(x,y)\cos(kz)$

$$\Rightarrow k = \frac{p\pi}{d}$$

Resonant cavity



$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le d$$

$$k^2 = \left(\frac{p\pi}{d}\right)^2 = \mu\varepsilon\omega^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$
$$\Rightarrow \omega^2 = \frac{1}{\mu\varepsilon}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$