PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 23:
Continue reading Chap. 11 –
Theory of Special Relativity
A. Lorentz transformation relations
B. Energy and momentum
C. Electromagnetic field transformations

WPU Physics Colloquium
TITLE: Measuring Molecular Forces Across Specific Proteins in Living Cells
SPEAKER: Professor Brian D. Hoffman,
Department of Biomedical Engineering,
Duke University
TIME: Wednesday March 26, 2014 at 4:00 PM
PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT
In vivo, cells adhere to the deformable extracellular matrix (ECM), that is both a source of applied force and a means of mechanical support. Cells detect and interpret mechanical signals, such as force and rigidity, from the ECM through mechanotransduction, while the connections between cells and the ECM, mediated by structure called focal adhesions. However, primary determinants of mechanotransduction, the molecular mechanisms mediating this process are largely unknown. Progress has been limited by an inability to measure dynamic forces across proteins in living cells. Therefore, we developed an experimentally calibrated Förster resonance energy transfer (FRET)-based biomarker that measures forces across specific proteins with piconewton sensitivity. The sensor has been validated in vitro, in a biological system, and in the nanoenvironment between the intact antibodies.
Lorentz transformations

Convenient notation:

\[ \beta = \frac{v}{c} \]
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]

Stationary frame \quad Moving frame

\[ ct = \gamma (ct' + \beta x') \]
\[ x = \gamma (x' + \beta v') \]
\[ y = y' \]
\[ z = z' \]

Lorentz transformations -- continued

For the moving frame with \( v = v \mathbf{\hat{k}} \):

\[ \mathbf{L} = \begin{pmatrix} \gamma & \gamma \beta & 0 & 0 \\ \gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \mathbf{L}^{-1} = \begin{pmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathbf{L} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \]

\[ \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathbf{L}^{-1} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \]

Notice:

\[ c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2 \]
Lorentz transformation of the velocity

Define:
\[ u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}, \]
\[ u'_x = \frac{dx'}{dt'}, \quad u'_y = \frac{dy'}{dt'}, \quad u'_z = \frac{dz'}{dt'} \]

\[ u_x = \frac{u'_x + u}{\gamma}, \quad u_y = \frac{u'_y}{\gamma}, \quad u_z = \frac{u'_z}{\gamma} \]

Where: \[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \]

Note that the velocity components themselves do not obviously transform according to a Lorentz transformation.

Velocity transformations continued:

Consider:
\[ u_x = \frac{u'_x + v}{1 + v u'/c^2}, \quad u_y = \frac{u'_y}{\gamma}, \quad u_z = \frac{u'_z}{\gamma} \]

Note that:
\[ \gamma = \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1 + v u'/c^2}{\sqrt{1 - (u/c)^2}} \]

\[ \Rightarrow \gamma u_x = \gamma (u_x' + v u_y') = \gamma u_x' + \beta u_y' \]
\[ \Rightarrow \gamma u_y = \gamma u_y' \]
\[ \Rightarrow \frac{\gamma u_z}{\gamma} = \gamma u_z' \]

Velocity 4-vector:
\[ \begin{bmatrix} \gamma c \\ \gamma u_x' \\ \gamma u_y' \\ \gamma u_z' \end{bmatrix} = \begin{bmatrix} \gamma c \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} \]

Significance of 4-velocity vector:
\[ \begin{bmatrix} \gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{bmatrix} \]

Introduce the "rest" mass \( m \) of particle characterized by velocity \( u \):
\[ m = \gamma m c \]

Properties of energy-moment 4-vector:
\[ \begin{bmatrix} E \\ p_x \\ p_y \\ p_z \end{bmatrix} = \gamma \begin{bmatrix} E \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} \]

Note: \( E^2 - p^2 c^2 = E' c^2 - p'^2 c^2 \)
Properties of Energy-momentum 4-vector -- continued

\[ \begin{align*}
E &= \gamma mc^2 \\
p_x &= \gamma p_x c \\
p_y &= \gamma p_y c \\
p_z &= \gamma p_z c
\end{align*} \]

Note: \( E^2 - p^2 c^2 = \frac{(mc^2)^2}{1-v^2} \left( 1 - \frac{v_x}{c} \right) \left( 1 - \frac{v_y}{c} \right) \left( 1 - \frac{v_z}{c} \right) = (mc^2)^2 - p^2 c^2 \)

Notion of "rest energy": For \( p = 0 \), \( E = mc^2 \)

Define kinetic energy: \( E_k = E - mc^2 = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \)

Non-relativistic limit: If \( \frac{p}{mc} \ll 1 \), \( E_k = mc^2 \left( \frac{1}{\gamma} \left( \frac{p}{mc} \right)^2 - 1 \right) \)

Summary of relativistic energy relationships

\[ \begin{align*}
E &= \sqrt{p^2 c^4 + m^2 c^4} = \gamma mc^2 \\
\gamma &= \frac{1}{\sqrt{1 - v^2}}
\end{align*} \]

Check: \( \sqrt{p^2 c^4 + m^2 c^4} = mc^2 \sqrt{1 + \gamma^2} \left( \frac{1}{\gamma} - 1 \right) \)

Example: For an electron \( mc^2 = 0.5 \text{ MeV} \) for \( E = 200 \text{ GeV} \)

\[ \begin{align*}
\gamma &= \frac{E}{mc^2} = 4 \times 10^{-9} \\
\gamma - 1 &= \frac{1}{2 \gamma} = 1 \times 3 \times 10^{-12}
\end{align*} \]

Special theory of relativity and Maxwell’s equations

Continuity equation: \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \)

Lorentz gauge condition: \( \frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \)

Potential equations: \( \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi = 4\pi \rho \)

\( \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = 4\pi \mathbf{J} \)

Field relations: \( \mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \)

\( \mathbf{B} = \nabla \times \mathbf{A} \)
More 4-vectors:

Time and position:
\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
x'^t \\
x' \\
y' \\
z'
\end{pmatrix}
\]

Charge and current:
\[
\begin{pmatrix}
\eta p \\
J_x \\
J_y \\
J_z
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
J'^t \\
J'_x \\
J'_y \\
J'_z
\end{pmatrix}
\]

Vector and scalar potentials:
\[
\begin{pmatrix}
\Phi \\
A_x \\
A_y \\
A_z
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\Phi' \\
A'_x \\
A'_y \\
A'_z
\end{pmatrix}
\]

Lorentz transformations
\[
L = \begin{pmatrix}
\gamma & \gamma \beta & 0 & 0 \\
\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Time and space:
\[
x'^t = L x^t = L_{\alpha \beta} x^\beta
\]

Charge and current:
\[
J'^t = L J^t = L_{\alpha \beta} J^\beta
\]

Vector and scalar potential:
\[
A'^t = L A^t = L_{\alpha \beta} A^\beta
\]

4-vector relationships
\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}
\Leftrightarrow
\begin{pmatrix}
A^t \\
A^x \\
A^y \\
A^z
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
A'^t \\
A'^x \\
A'^y \\
A'^z
\end{pmatrix}
\]

Keeping track of signs - lower index 4-vector \( A_\alpha = (\xi^\alpha, -\xi^\alpha) \)

Derivative operators:
\[
\partial^\alpha = \left( \frac{\partial}{c \xi^\alpha}, -\frac{\partial}{c \xi^\alpha} \right)
\]

\[
\partial_\alpha = \left( \frac{\partial}{c \xi^\alpha}, \frac{\partial}{c \xi^\alpha} \right)
\]
Special theory of relativity and Maxwell’s equations

Continuity equation: \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \)  \( \hat{\partial}_x J^x = 0 \)

Lorentz gauge condition: \( \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \)  \( \hat{\partial}_x A^x = 0 \)

Potential equations:
1. Lorentz equation: \( \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi \rho \)
2. Continuity equation: \( \frac{1}{c^2} \frac{\partial^2 A^x}{\partial t^2} - \nabla^2 A^x = \frac{4\pi}{c} J^x \)

Field relations:
\( \mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \)
\( \mathbf{B} = \nabla \times \mathbf{A} \)

Electric and Magnetic field relationships

\( \mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \)
\( E_x = \frac{\partial \Phi}{\partial x} - \frac{\partial A_y}{\partial t} \)
\( E_y = \frac{\partial \Phi}{\partial y} - \frac{\partial A_x}{\partial t} \)
\( E_z = \frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{\partial t} \)
\( \mathbf{B} = \nabla \times \mathbf{A} \)
\( B_x = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \)
\( B_y = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \)
\( B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \)

Field strength tensor
\( F^{\alpha \beta} = \left( \begin{array}{ccc} 0 & -E_x & -E_y \\ E_x & 0 & -B_z \\ -E_y & B_z & 0 \end{array} \right) \)

Transformation of field strength tensor
\( F^{\alpha \beta} = L^{\alpha \beta}_{\gamma \delta} F^{\gamma \delta} \)

\( L^{\alpha \beta} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \)

Special transformation:
\( F^{\alpha \beta} = \left( \begin{array}{ccc} 0 & -B_x & -B_y \\ -E_x & 0 & -E_z \\ -E_y & E_z & 0 \end{array} \right) \)

\( F^{\alpha \beta} = \left( \begin{array}{ccc} \gamma (E_x^0 - B_x B_z) & \gamma (B_z^0 - E_x B_x) & \gamma (E_x^0 + B_x B_z) \\ \gamma (E_x^0 + B_x B_z) & \gamma (B_z^0 - E_x B_x) & \gamma (E_x^0 - B_x B_z) \\ \gamma (E_x^0 - B_x B_z) & \gamma (E_x^0 + B_x B_z) & E_x' \end{array} \right) \)
Lorentz transformation of electromagnetic fields:

\begin{align*}
E_x &= E'_x, \\
E_y &= \gamma (E'_x + \beta B'_y), \\
E_z &= \gamma (E'_x - \beta B'_y), \\
B_x &= B'_x, \\
B_y &= \gamma (B'_y - \beta E'_x), \\
B_z &= \gamma (B'_y + \beta E'_x).
\end{align*}