PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 27:

Continue reading Chap. 14 – Synchrotron radiation

1. Radiation from electron synchrotron devices

2. Radiation from astronomical objects in circular orbits

Radiation from charged particle in circular path

$$\frac{dP_r(t_v)}{d\Omega} = \frac{q^2}{4\pi c} \left| \mathbf{R} \times \left[ \mathbf{R} - \beta \mathbf{R} \times \beta \mathbf{R} \right] \right|$$

$$= \frac{q^2}{4\pi c} \left| \beta \mathbf{R} - \left( \mathbf{R} \beta \mathbf{R} \right) \left( 1 - \beta^2 \right) \right|$$

$$P_r(t_v) = \left[ d\Omega \frac{dP_r(t_v)}{d\Omega} \right] = \frac{q^2}{2\pi c^2} |v|^4$$
Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral:

\[ \frac{\partial^2 I}{\partial \omega^2 \Omega} = \frac{q^2 \omega^3}{4\pi^2 c} \int_{-\infty}^{\infty} dt \left( \mathbf{r} \times \mathbf{B}(t) \right) e^{i\omega(t_i - t) \mathbf{r} \cdot \mathbf{r}(t) / c} \]

Synchrotron radiation light source installations

Synchrotron radiation center in Madison, Wisconsin

E₀ = 0.5 GeV and 1 GeV; \( \lambda_c = 20 \text{ Å} \) and 10 Å

http://www.src.wisc.edu/
E = 0.6 GeV; $\lambda = 20$ Å

http://www.bnl.gov/ps/

Spectral intensity relationship:

$$\frac{d^2 I}{d\Omega} = \frac{q^2 m^2}{4\pi^2} \int d^2 k [\mathbf{e}(\mathbf{r}) \cdot \mathbf{e}(\mathbf{r}')] e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}$$

For convenience, choose:

$$\mathbf{r} = \hat{x} \cos \theta + \hat{z} \sin \theta$$

Note that we have previously shown that in the radiation zone, the Poynting vector is in the $\mathbf{r}$ direction; we can then choose to analyze two orthogonal polarizations on directions:

$$\mathbf{e}_1 = \hat{y}$$
$$\mathbf{e}_2 = -\hat{x} \sin \theta + \hat{z} \cos \theta$$

$\mathbf{r} \times (\mathbf{e} \times \mathbf{B}) = \beta (-\mathbf{e}_1 \sin (vt/\rho) + \mathbf{e}_2 \sin \theta \cos (vt/\rho))$
We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v = (1 - 1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta = 0$, and the relevant integration times $t$ are close to $\tau = 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c = 3\gamma q^2 c^2$.

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{3\gamma q^2}{4\pi^2 c^2} \left( \frac{m\omega_c}{\omega} \right)^2 (1 + \gamma'\omega')^{\gamma'} \left[ K_{\nu_+} \left( \frac{m\omega_c}{2\gamma} (1 + \gamma'\omega')^{\gamma'} \right) \right] \]

Some details:

**Modified Bessel functions**

\[ K_{+\nu}(\xi) = \sqrt{\frac{\pi}{\xi}} \int_0^\infty e^{-\xi u (1 + \frac{1}{2} u^2)} \frac{\sin(\nu u)}{u} \, du \]

**Exponential factor**

\[ e^{i\nu_+ \mathbf{r} \cdot \mathbf{R}(t)} = e^{i\nu \mathbf{r} \cdot \mathbf{R}(t)} - \frac{\mathbf{r} \cdot \mathbf{R}(t)}{c} \cos \theta \sin(v t / \rho) \]

In the limit of $t \to 0$, $\theta = 0$, $v = 1 - \frac{1}{2\gamma^2}$,

\[ e^{i\nu_+ \mathbf{r} \cdot \mathbf{R}(t)} = \frac{e^{i\nu t}}{2\gamma^2} \left( 1 + \gamma^2\omega^2 \right) + \frac{e^{i\nu \mathbf{r} \cdot \mathbf{R}(t)}}{6\rho^2} - \frac{3}{2} e^{i\nu \mathbf{r} \cdot \mathbf{R}(t)} + \frac{1}{3} e^{i\nu \mathbf{r} \cdot \mathbf{R}(t)} \]

where $\xi = \frac{e^{i\nu \mathbf{r} \cdot \mathbf{R}(t)}}{3\gamma^2} \left( 1 + \gamma^2\omega^2 \right)^{\frac{3}{2}}$ and $x = \frac{e^{i\nu t}}{\rho^4} \left( 1 + \gamma^2\omega^2 \right)^{\frac{3}{2}}$.
By plotting the intensity as a function of , we see that the intensity is largest near . The plot below shows the intensity as a function of for = 0, 0.5 and 1:

\[
\frac{d^2I}{d\omega d\Omega} = \frac{3g^2y^2}{4\pi c^2} \left(1 + y^2 \theta^2\right)^2 \left[K_{\frac{1}{2}} \left(\frac{\omega}{2\omega_c} \left(1 + y^2 \theta^2\right)^{\frac{3}{2}}\right)\right]
\]

The second example of synchrotron radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of . For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

\[
e^{i\omega t} = \sum_{m=-\infty}^{\infty} J_m(\alpha) e^{im\theta}
\]

Here is a Bessel function of integer order . In our case \(\alpha = \frac{vt}{\rho}\) and \(\omega = \frac{v}{\rho}\).

\[
C_i(\omega) = \int_0^\infty dt \sin(\omega t / \rho) e^{i\alpha \cos \theta} = \frac{\alpha}{\omega} e^{i\alpha \cos \theta} \int_0^\infty dt e^{i\alpha \cos \theta}
\]

Astronomical synchrotron radiation -- continued:

Note that:

\[
\int_{-\infty}^\infty dt e^{i\omega t + i\alpha \cos \theta} = 2\pi \delta(\omega - m \rho).
\]

\[
\Rightarrow C_i(\omega) = 2\pi \sum_{m=-\infty}^{\infty} J_m\left(\frac{\alpha}{c} \cos \theta\right) \delta(\omega - m \rho),
\]

where \(J_m(\alpha) = \frac{dJ_m(\alpha)}{d\alpha}\).

Similarly:

\[
C_i(\omega) = \int_0^\infty d\omega' \sin \theta \cos(\omega' t / \rho) e^{i\alpha \cos \theta} = 2\pi \tan \theta e^{i\alpha \cos \theta} \sum_{m=-\infty}^{\infty} J_m\left(\frac{\alpha}{c} \cos \theta\right) \delta(\omega - m \rho).
\]
Astronomical synchrotron radiation -- continued:
In both of the expressions, the sum over $m$ includes both negative and positive values. However, only the positive values of $\omega$ and therefore positive values of $m$ are of interest. Using the identity $J_{\omega m}(\omega) = (-1)^m J_{\omega m}(\omega)$, the result becomes:

$$\frac{d^3J}{d\omega d\Omega} = \frac{q^3}{c} \sum_{m} \delta(\omega - m \frac{v}{\rho}) \left[ J_{\omega m} \left( \frac{e\rho}{c} \cos \theta \right) \right] + \frac{\tan^4 \theta}{v^4 / c^4} \left[ J_{\omega m} \left( \frac{e\rho}{c} \cos \theta \right) \right].$$

These results were derived by Julian Schwinger (Phys. Rev. 75, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. For information on man-made synchrotron sources, the following web page is useful: