PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

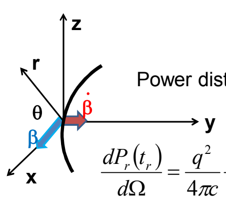
Plan for Lecture 27:

Continue reading Chap. 14 – Synchrotron radiation

1. **Radiation from electron synchrotron devices**
2. **Radiation from astronomical objects in circular orbits**

	Fri: 03/14/2014	*Spring Break*			
19	Mon: 03/17/2014	Chap. 8	Wave guides;Take-home exam due	#18	3/21/2014
20	Wed: 03/19/2014	Chap. 9	Sources of Electromagnetic Waves	#19	3/21/2014
21	Fri: 03/21/2014	Chap. 9	Sources of Electromagnetic Waves	#20	3/28/2014
22	Mon: 03/24/2014	Chap. 11	Special Theory of Relativity	#21	3/28/2014
23	Wed: 03/26/2014	Chap. 11	Special Theory of Relativity	#22	3/28/2014
24	Fri: 03/28/2014	Chap. 11	Special Theory of Relativity	#23	4/04/2014
25	Mon: 03/31/2014	Chap. 14	Radiation from moving charges	#24	4/04/2014
26	Wed: 04/02/2014	Chap. 14	Radiation from moving charges	#25	4/04/2014
27	Fri: 04/04/2014	Chap. 14	Radiation from moving charges	#26	4/11/2014
28	Mon: 04/07/2014				
29	Wed: 04/09/2014				
30	Fri: 04/11/2014				
31	Mon: 04/14/2014				
32	Wed: 04/16/2014				
	Fri: 04/18/2014	Good Friday			
33	Mon: 04/21/2014				
34	Wed: 04/23/2014				
35	Fri: 04/23/2014				
	Mon: 04/28/2014		Presentations Part I		
	Wed: 04/30/2014		Presentations Part II		

Radiation from charged particle in circular path



Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

$$= \frac{q^2}{4\pi c}\frac{\left|\dot{\boldsymbol{\beta}}\right|^2\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2-\left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2\left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

$$P_r(t_r) = \int d\Omega\,\frac{dP_r(t_r)}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2\gamma^4$$

Spectral composition of electromagnetic radiation -- continued

When the dust clears, the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right] e^{i\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

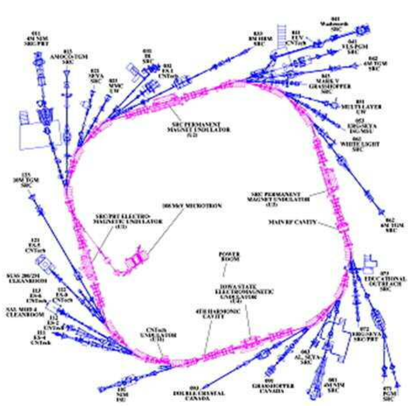
Synchrotron radiation light source installations

Synchrotron radiation center in Madison, Wisconsin

E_c=0.5 GeV and 1 GeV; λ_c = 20 Å and 10 Å

http://www.src.wisc.edu/

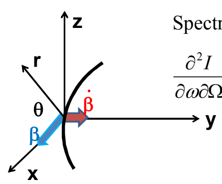
SRC – "Aladdin" -- Madison Wisconsin



Brookhaven National Laboratory – National Light Source

E_c=0.6 GeV; λ_c = 20 Å

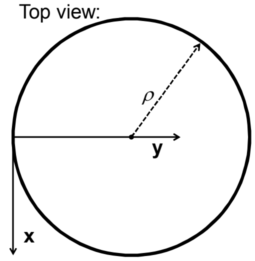
http://www.bnl.gov/ps/



Spectral intensity relationship:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

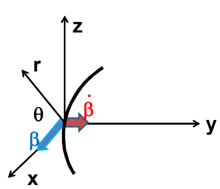
Top view:



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho)$$
$$+\ \rho\hat{\mathbf{y}}(1-\cos(vt_r/\rho))$$
$$\boldsymbol{\beta}(t_r) = \beta(\hat{\mathbf{x}}\cos(vt_r/\rho)+\hat{\mathbf{y}}\sin(vt_r/\rho))$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$



$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho)$$
$$+\ \rho\hat{\mathbf{y}}(1-\cos(vt_r/\rho))$$
$$\boldsymbol{\beta}(t_r) = \beta(\hat{\mathbf{x}}\cos(vt_r/\rho)+\hat{\mathbf{y}}\sin(vt_r/\rho))$$

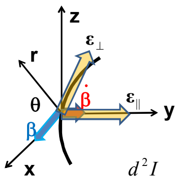
For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$

Note that we have previous shown that in the radiation zone, the Poynting vector is in the $\hat{\mathbf{r}}$ direction; we can then choose to analyze two orthogonal polarizati on directions :

$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho)+\boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$



$$\boldsymbol{\varepsilon}_\parallel = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_\perp = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_\parallel \sin(vt_r/\rho) + \boldsymbol{\varepsilon}_\perp \sin\theta\cos(vt_r/\rho)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\beta)e^{i\omega(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)}dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_\parallel(\omega)|^2 + |C_\perp(\omega)|^2\right\}$$

$$C_\parallel(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_\perp(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

We will analyze this expression for two different cases. The first case, is appropriate for man-made synchrotrons used as light sources. In this case, the light is produced by short bursts of electrons moving close to the speed of light ($v \approx c(1-1/(2\gamma^2))$) passing a beam line port. In addition, because of the design of the radiation ports, $\theta \approx 0$, and the relevant integration times t are close to $t \approx 0$. This results in the form shown in Eq. 14.79 of your text. It is convenient to rewrite this form in terms of a critical frequency $\omega_c \equiv \frac{3c\gamma^3}{2\rho}$.

$$\frac{d^2I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right.$$

Some details:

Modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3}\int_0^\infty dx \cos\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi) = \sqrt{3}\int_0^\infty dx\, x \sin\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)\right) = \omega\left(t_r - \frac{\rho}{c}\cos\theta\sin(vt_r/\rho)\right)$$

In the limit of $t_r \approx 0,\ \theta \approx 0,\ v \approx c\left(1-\frac{1}{2\gamma^2}\right)$

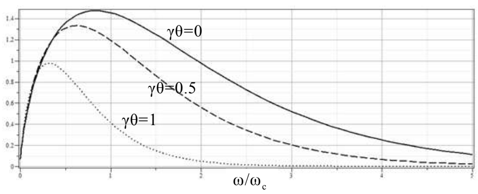
$$\omega\left(t_r - \hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)\right) \approx \frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right) + \frac{\omega c^2 t_r^3}{6\rho^2} = \frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)$$

where $\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$ and $x = \frac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

$$\frac{d^2I}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2 (1+\gamma^2\theta^2)^2 \left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}(1+\gamma^2\theta^2)^{\frac{3}{2}}\right)\right]^2\right.$$

By plotting the intensity as a function of ω, we see that the intensity is largest near $\omega \approx \omega_c$. The plot below shows the intensity as a function of ω/ω_c for $\gamma\theta=0$, 0.5 and 1:

$\frac{d^2I}{d\omega d\Omega}$



The second example of synchroton radiation comes from a distant charged particle moving in a circular trajectory such that the spectrum represents a superposition of light generated over many complete circles. In this case, there is an interference effect which results in the spectrum consisting of discrete multiples of v/ρ. For this case we need to reconsider the analysis. There is a very convenient Bessel function identity of the form:

$$e^{-ia\sin\alpha} = \sum_{m=-\infty}^{\infty} J_m(a)e^{-im\alpha}$$ Here $J_m(a)$ is a Bessel function of integer order m.

In our case $a = \frac{\omega\rho}{c}\cos\theta$ and $\alpha = \frac{vt}{\rho}$.

$$C_{\|}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))} = \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\int_{-\infty}^{\infty} dt e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= \frac{c}{-i\omega\rho}\frac{\partial}{\partial\cos\theta}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)2\pi\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

Note that:

$$\int_{-\infty}^{\infty} dt e^{i(\omega - m\frac{v}{\rho})t} = 2\pi\delta(\omega - m\frac{v}{\rho}).$$

$$\Rightarrow C_{\|}(\omega) = 2\pi i \sum_{m=-\infty}^{\infty} J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}),$$

where $J_m'(a) \equiv \frac{dJ_m(a)}{da}$

Similarly:

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin\theta\cos(vt/\rho) e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$= 2\pi\frac{\tan\theta}{v/c}\sum_{m=-\infty}^{\infty} J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\delta(\omega - m\frac{v}{\rho}).$$

Astronomical synchrotron radiation -- continued:

In both of the expressions, the sum over *m* includes both negative and positive values. However, only the positive values of ω and therefore positive values of *m* are of interest. Using the identity: $J_{-m}(a) = (-1)^m J_m(a),$ the result becomes:

$$\frac{d^2 I}{d\omega d\Omega} =$$

$$\frac{q^2\omega^2\beta^2}{c}\sum_{m=0}^{\infty}\delta(\omega - m\frac{v}{\rho})\left\{\left[J_m'\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2 + \frac{\tan^2\theta}{v^2/c^2}\left[J_m\left(\frac{\omega\rho}{c}\cos\theta\right)\right]^2\right\}.$$

These results were derived by Julian Schwinger (Phys. Rev. **75**, 1912-1925 (1949)). The discrete case is similar to the result quoted in Problem 14.15 in Jackson's text. For information on man-made synchrotron sources, the following web page is useful:
http://www.als.lbl.gov/als/synchrotron_sources.html.