

**PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107**

Plan for Lecture 28:
Continue reading Chap. 14 –

Review of radiation from charged particles

1. Review of power distribution and spectral and polarization properties
2. Thompson and Compton scattering

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Wed: 03/12/2014	Spring Break
Fri: 03/14/2014	Spring Break
19 Mon: 03/17/2014	Chap. 8 Wave guides;Take-home exam due #18 3/21/2014
20 Wed: 03/19/2014	Chap. 9 Sources of Electromagnetic Waves #19 3/21/2014
21 Fri: 03/21/2014	Chap. 9 Sources of Electromagnetic Waves #20 3/28/2014
22 Mon: 03/24/2014	Chap. 11 Special Theory of Relativity #21 3/28/2014
23 Wed: 03/26/2014	Chap. 11 Special Theory of Relativity #22 3/28/2014
24 Fri: 03/28/2014	Chap. 11 Special Theory of Relativity #23 4/04/2014
25 Mon: 03/31/2014	Chap. 14 Radiation from moving charges #24 4/04/2014
26 Wed: 04/02/2014	Chap. 14 Radiation from moving charges #25 4/04/2014
27 Fri: 04/04/2014	Chap. 14 Radiation from moving charges #26 4/11/2014
28 Mon: 04/07/2014	Chap. 14 Radiation from moving charges #27 4/11/2014
29 Wed: 04/09/2014	
30 Fri: 04/11/2014	
31 Mon: 04/14/2014	
32 Wed: 04/16/2014	
Fri: 04/18/2014	Good Friday
33 Mon: 04/21/2014	
34 Wed: 04/23/2014	
35 Fri: 04/25/2014	
Mon: 04/28/2014	Presentations Part I
Wed: 04/30/2014	Presentations Part II

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WAKE FOREST UNIVERSITY Department of Physics

News

-  Prof. Carroll named APS Fellow
-  Protein research led by Prof. Che featured in news
-  Prof. Thonhauser receives Award for Excellence in Research
-  Peter Diemer and Prof. Jurchescu receive Wake Forest Innovation Award

Events

- Wed, Apr. 9, 2014** Diamond-like semiconductors
Prof. Althen, Duquesne
4:00 PM in Olin 101
Refreshments at 3:30 PM in Olin Lobby
- Wed, Apr. 16, 2014** Organic electronics
Prof. Conrad, App. State
4:00 PM in Olin 101
Refreshments at 3:30 PM in Olin Lobby

Profiles in Physics

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Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v} R}{c} \right) \left(1 - \frac{v^2}{c^2} \right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v} R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$$

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Electric and magnetic fields far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}\right)^3} \left\{ \mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

$$\text{Let } \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \quad \beta \equiv \frac{\mathbf{v}}{c} \quad \dot{\beta} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \beta \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

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Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \beta \cdot \hat{\mathbf{R}})^3} \{ \hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}] \}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}]|^2}{(1 - \beta \cdot \hat{\mathbf{R}})^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}]|^2}{(1 - \beta \cdot \hat{\mathbf{R}})^6}$$

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Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}]|^2}{(1 - \beta \cdot \hat{\mathbf{R}})^6} \Big|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time t . We are often interested in the power per unit retarded time $t_r = t - R/c$:

$$\frac{dP(t)}{d\Omega} = \frac{dP_r(t_r)}{d\Omega} \frac{dt_r}{dt} = 1 - \beta \cdot \hat{\mathbf{R}}$$

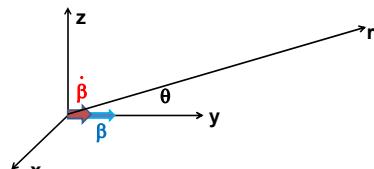
$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \beta) \times \dot{\beta}]|^2}{(1 - \beta \cdot \hat{\mathbf{R}})^6} \Big|_{t_r = t - R/c}$$

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Power distribution for linear acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\beta})|^2}{(1 - \beta \cdot \hat{\mathbf{R}})^5} \Big|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

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Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\dot{\beta}|^2 \left(1 - \beta \cdot \hat{R}\right)^2 - \left(\hat{R} \cdot \dot{\beta}\right)^2 \left(1 - \beta^2\right)}{\left(1 - \beta \cdot \hat{R}\right)^5}_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{\left(1 - \beta \cos(\theta)\right)^3} \left(1 - \frac{\cos^2 \theta \sin^2 \phi}{r^2 \left(1 - \beta \cos(\theta)\right)^2}\right)$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

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Spectral composition of electromagnetic radiation
 Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude :

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \quad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Note that: $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\Rightarrow \frac{dW}{d\Omega} = \int_0^{\infty} d\omega \left(|\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_0^{\infty} d\omega \frac{\partial^2 I}{\partial \Omega \partial \omega}$$

$$\frac{\partial^2 I}{\partial \Omega \partial \omega} \equiv 2 |\tilde{\mathbf{a}}(\omega)|^2$$

Spectral composition of electromagnetic radiation -- continued

The spectral intensity therefore depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^3}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{B}(t_r)) \right] e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Example for circular motion:

The figure shows a 3D coordinate system with axes x, y, and z. A point on a circular path is shown with position vector \mathbf{r} at an angle θ from the x-axis. The velocity vector \mathbf{v}_r is tangent to the circle, and the acceleration vector $\mathbf{R}_q(t_r)$ is directed towards the center. The angle θ is measured from the positive x-axis.

Top view:

The top view shows a circle centered at the origin. A radius vector ρ is drawn from the center to a point on the circle, representing the radial distance. The horizontal axis is labeled **y**, and the vertical axis is implied by the z-axis.

$\mathbf{R}_q(t_r) = \rho \hat{\mathbf{x}} \sin(v_{t_r} / \rho) + \rho \hat{\mathbf{y}} (1 - \cos(v_{t_r} / \rho))$

$\mathbf{p}(t_r) = \rho (\hat{\mathbf{x}} \cos(v_{t_r} / \rho) + \hat{\mathbf{y}} \sin(v_{t_r} / \rho))$

For convenience, choose:
 $\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos \theta + \hat{\mathbf{z}} \sin \theta$

$\mathbf{r} \times \mathbf{B} = \mathbf{\hat{y}} \times (\mathbf{\hat{r}} \times \mathbf{B}) = \mathbf{\hat{z}} \sin \theta + \mathbf{-\hat{x}} \cos \theta$

$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \mathbf{\beta}) e^{i\omega(t - \mathbf{\hat{r}} \cdot \mathbf{R}_q(t)/c)} dt \right|^2$

$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2 \omega^2 \beta^2}{4\pi^2 c} \{ |C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2 \}$

$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt \sin(vt/\rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))}$

$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt \sin \theta \cos(vt/\rho) e^{i\omega(t - \frac{\rho}{c} \cos \theta \sin(vt/\rho))}$

Synchrotron radiation geometry –
using modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3} \int_0^\infty dx \cos\left[\frac{1}{2}\xi\left(x + \frac{1}{3}x^3\right)\right] \quad K_{2/3}(\xi) = \sqrt{3} \int_0^\infty dx x \sin\left[\frac{1}{2}\xi\left(x + \frac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \frac{\hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)}{c}\right) = \omega\left(t_r - \frac{\rho}{c} \cos\theta \sin(vt_r/\rho)\right)$$

In the limit of $t_r \approx 0, \theta \approx 0, v \approx c$

$$\left(1 - \frac{1}{2\gamma^2}\right)$$

$$\omega\left(t_r - \frac{\hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)}{c}\right) \approx \frac{\omega t_r}{2\gamma^2} (1 + \gamma^2 \theta^2) + \frac{\omega c^2 t_r^3}{6\rho^2} = \frac{3}{2} \xi \left(x + \frac{1}{3}x^3\right)$$

where $\xi = \frac{\omega\rho}{3c\gamma^3} (1 + \gamma^2 \theta^2)^{3/2}$ and $x = \frac{c\gamma t_r}{\rho(1 + \gamma^2 \theta^2)^{1/2}}$

Spectral form of synchrotron radiation in this case:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{3q^2 \gamma^2}{4\pi^2 c} \left(\frac{\omega}{\omega_c} \right)^2 (1 + \gamma^2 \theta^2)^2 \left\{ \left[K_{2/3} \left(\frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} \left[K_{1/3} \left(\frac{\omega}{2\omega_c} (1 + \gamma^2 \theta^2)^{3/2} \right) \right]^2 \right\}$$

$$\omega_c \equiv \frac{3c\gamma^3}{2\rho}$$

At $\theta = 0$:

$$\text{note that for } \omega \ll \omega_c \Rightarrow \frac{\partial^2 I}{\partial \omega \partial \Omega} \approx \frac{q^2}{\pi^2 c} (\Gamma(\frac{2}{3}))^2 \left(\frac{3\omega^2 \rho^2}{4c^2} \right)^{1/3}$$

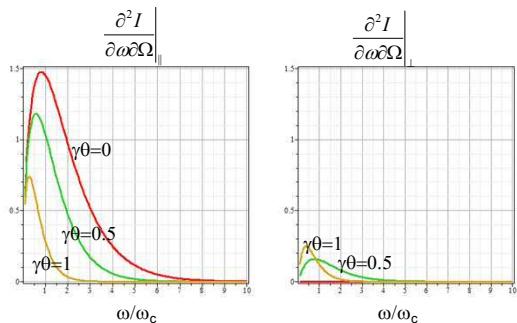
$$\text{and for } \omega \gg \omega_c \Rightarrow \frac{\partial^2 I}{\partial \omega \partial \Omega} \approx \frac{3q^2}{4\pi c} \gamma^2 \left(\frac{\omega}{\omega_c} \right)^2 e^{-\omega/\omega_c}$$

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Plots of the intensity function for synchrotron radiation

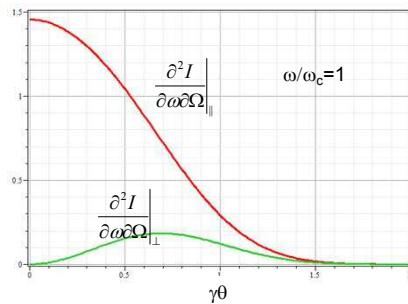


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Plots of the intensity function for synchrotron radiation



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Thompson scattering – non relativistic approximation

Power radiated in direction \hat{r} by charged particle with acceleration $\dot{\mathbf{v}}$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \dot{\mathbf{v}})|^2$$

Suppose that the acceleration $\dot{\mathbf{v}}$ of a particle (charge q and mass m_q) is caused by an electric field: $\mathbf{E}(\mathbf{r}, t) = \Re(\epsilon_0 E_0 e^{i\mathbf{k}_0 \cdot \mathbf{r} - i\omega t})$

$$\dot{\mathbf{v}} = \frac{q}{m_q} \Re(\epsilon_0 E_0 e^{i\mathbf{k}_0 \cdot \mathbf{r} - i\omega t})$$

$$\text{Time averaged power: } \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{q^2}{m_q c^2} \right)^2 |E_0|^2 |\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \epsilon_0)|^2$$

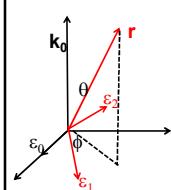
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Thompson scattering – non relativistic approximation -- continued

$$\text{Time averaged power: } \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{q^2}{m_q c^2} \right)^2 |E_0|^2 |\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \epsilon_0)|^2$$



Polarization of incident light: $\epsilon_0 = \hat{x}$

Polarization of scattered light:

$$\begin{aligned}\epsilon_1 &= \cos \theta (\hat{x} \cos \phi + \hat{y} \sin \phi) - \hat{z} \sin \theta \\ \epsilon_2 &= -\hat{x} \sin \phi + \hat{y} \cos \phi\end{aligned}$$

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Thompson scattering – non relativistic approximation -- continued

Time averaged power with polarization ϵ^* :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{q^2}{m_q c^2} \right)^2 |E_0|^2 |\epsilon^* \cdot \epsilon_0|^2$$

Scattered light may be polarized parallel to incident field or polarized with an angle θ so that the time and polarization averaged cross section is given by:

$$\langle |\epsilon^* \cdot \epsilon_0|^2 \rangle_\phi = \langle |\epsilon_1 \cdot \epsilon_0|^2 \rangle_\phi + \langle |\epsilon_2 \cdot \epsilon_0|^2 \rangle_\phi = \frac{1}{2} \cos^2 \theta + \frac{1}{2}$$

$$\text{Averaged cross section: } \left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2} \right)^2 \frac{1}{2} (1 + \cos^2 \theta)$$

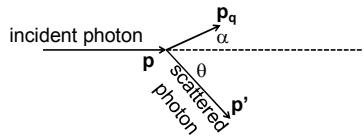
This formula is appropriate in the X-ray scattering of electrons or soft γ -ray scattering of protons

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Thompson scattering – relativistic and quantum modifications



Conservation of momentum and energy :

$$p = p' \cos \theta + p_q \cos \alpha \quad pc = \hbar \omega$$

$$0 = p' \sin \theta - p_q \sin \alpha \quad p' c = \hbar \omega'$$

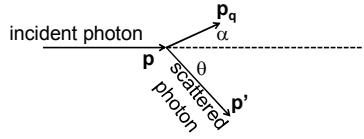
$$\hbar\omega + m_q c^2 = \hbar\omega' + \sqrt{p_q^2 c^2 + (m_q c^2)^2}$$

$$\frac{p'}{p} = \frac{1}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}$$

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Thompson scattering – relativistic and quantum modifications



Relativistic and quantum modifications to averaged cross section:

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2} \right)^2 \left(\frac{p'}{p} \right)^2 \frac{1}{2} (1 + \cos^2 \theta)$$

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