PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 28:

Continue reading Chap. 14 –

Review of radiation from charged particles

1. **Review of power distribution and spectral and polarization properties**
2. **Thompson and Compton scattering**

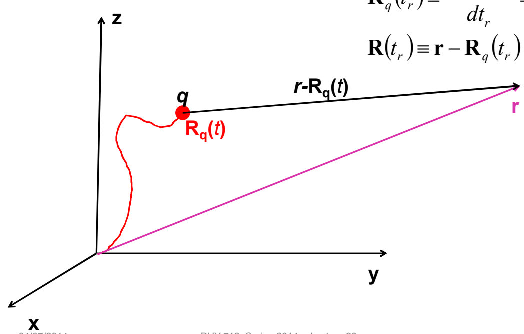
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
19	Mon: 03/17/2014	Chap. 8	Wave guides;Take-home exam due	#18	3/21/2014
20	Wed: 03/19/2014	Chap. 9	Sources of Electromagnetic Waves	#19	3/21/2014
21	Fri: 03/21/2014	Chap. 9	Sources of Electromagnetic Waves	#20	3/28/2014
22	Mon: 03/24/2014	Chap. 11	Special Theory of Relativity	#21	3/28/2014
23	Wed: 03/26/2014	Chap. 11	Special Theory of Relativity	#22	3/28/2014
24	Fri: 03/28/2014	Chap. 11	Special Theory of Relativity	#23	4/04/2014
25	Mon: 03/31/2014	Chap. 14	Radiation from moving charges	#24	4/04/2014
26	Wed: 04/02/2014	Chap. 14	Radiation from moving charges	#25	4/04/2014
27	Fri: 04/04/2014	Chap. 14	Radiation from moving charges	#26	4/11/2014
28	Mon: 04/07/2014	Chap. 14	Radiation from moving charges	#27	4/11/2014
29	Wed: 04/09/2014				
30	Fri: 04/11/2014				
31	Mon: 04/14/2014				
32	Wed: 04/16/2014				
	Fri: 04/18/2014	Good Friday			
33	Mon: 04/21/2014				
34	Wed: 04/23/2014				
35	Fri: 04/23/2014				
	Mon: 04/28/2014		Presentations Part I		
	Wed: 04/30/2014		Presentations Part II		

Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$



Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \qquad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \qquad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^{\,2}}$$

Electric and magnetic fields far from source:

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}$$

Let $\hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \qquad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \qquad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{cR\left(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

Poynting vector:

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{cR\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^3}\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right\}$$

$$\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}(\mathbf{r},t)=\frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}(\mathbf{r},t)\right|^2=\frac{q^2}{4\pi cR^2}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

$$\frac{dP}{d\Omega}=\mathbf{S}\cdot\hat{\mathbf{R}}R^2=\frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}$$

Radiation distribution in the relativistic case

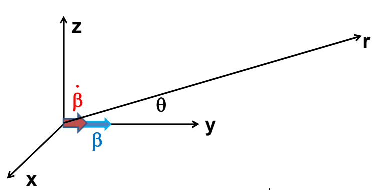
$$\frac{dP}{d\Omega}=\mathbf{S}\cdot\hat{\mathbf{R}}R^2=\frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^6}\right|_{t_r=t-R/c}$$

This expression gives us the energy per unit field time *t*. We are often interested in the power per unit retarded time $t_r=t-R/c$:

$$\frac{dP(t)}{d\Omega}=\frac{dP_r(t_r)}{d\Omega}\frac{dt_t}{dt}\qquad\frac{dt}{dt_r}=1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega}=\frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c}$$

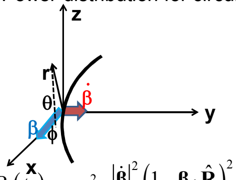
Power distribution for linear acceleration



$$\frac{dP_r(t_r)}{d\Omega}=\frac{q^2}{4\pi c}\left.\frac{\left|\hat{\mathbf{R}}\times\left(\hat{\mathbf{R}}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\right|_{t_r=t-R/c}=\frac{q^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2\frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}$$

$$P_r(t_r)=\int\frac{dP_r(t_r)}{d\Omega}d\Omega=\frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2\gamma^6\qquad\text{where }\gamma\equiv\frac{1}{\sqrt{1-\beta^2}}$$

Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c}\frac{\left|\dot{\boldsymbol{\beta}}\right|^2\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^2 - \left(\hat{\mathbf{R}}\cdot\dot{\boldsymbol{\beta}}\right)^2\left(1-\beta^2\right)}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}}\right)^5}\Bigg|_{t_r=t-R/c}$$

$$= \frac{q^2}{4\pi c^3}\frac{\left|\dot{\mathbf{v}}\right|^2}{\left(1-\beta\cos(\theta)\right)^3}\left(1-\frac{\cos^2\theta\sin^2\phi}{\gamma^2\left(1-\beta\cos(\theta)\right)^2}\right)$$

$$P_r(t_r) = \int d\Omega\frac{dP_r(t_r)}{d\Omega} = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2\gamma^4$$

Spectral composition of electromagnetic radiation

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt\frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt\left|\boldsymbol{a}(t)\right|^2 = \int_{-\infty}^{\infty} d\omega\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Fourier amplitude :

$$\tilde{\boldsymbol{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\,\boldsymbol{a}(t)e^{i\omega t} \qquad \boldsymbol{a}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,\tilde{\boldsymbol{a}}(\omega)e^{-i\omega t}$$

Note that: $\tilde{\boldsymbol{a}}(\omega) = \tilde{\boldsymbol{a}}^*(-\omega)$

$$\Rightarrow \frac{dW}{d\Omega} = \int_0^{\infty} d\omega\left(\left|\tilde{\boldsymbol{a}}(\omega)\right|^2 + \left|\tilde{\boldsymbol{a}}(-\omega)\right|^2\right) \equiv \int_0^{\infty} d\omega\frac{\partial^2 I}{\partial\Omega\partial\omega}$$

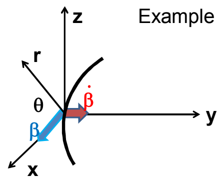
$$\frac{\partial^2 I}{\partial\Omega\partial\omega} \equiv 2\left|\tilde{\boldsymbol{a}}(\omega)\right|^2$$

Spectral composition of electromagnetic radiation -- continued

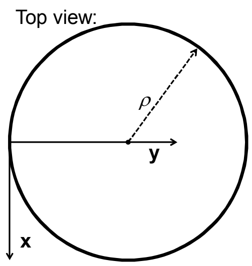
The spectral intensity therefore depends on the following integral:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\boldsymbol{\beta}(t_r)\right)\right]e^{i\omega\left(t_r-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)/c\right)}\right|^2$$

Example for circular motion:



Top view:

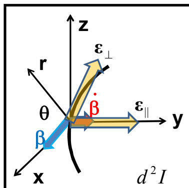


$$\mathbf{R}_q(t_r) = \rho\hat{\mathbf{x}}\sin(vt_r/\rho) + \rho\hat{\mathbf{y}}(1-\cos(vt_r/\rho))$$

$$\boldsymbol{\beta}(t_r) = \beta(\hat{\mathbf{x}}\cos(vt_r/\rho)+\hat{\mathbf{y}}\sin(vt_r/\rho))$$

For convenience, choose:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta$$



$$\boldsymbol{\varepsilon}_{\parallel} = \hat{\mathbf{y}} \qquad \boldsymbol{\varepsilon}_{\perp} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{\beta}) = \beta\left(-\boldsymbol{\varepsilon}_{\parallel}\sin(vt_r/\rho) + \boldsymbol{\varepsilon}_{\perp}\sin\theta\cos(vt_r/\rho)\right)$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\beta)e^{i\omega(t-\hat{\mathbf{r}}\cdot\mathbf{R}_q(t)/c)}dt\right|^2$$

$$\frac{d^2I}{d\omega d\Omega} = \frac{q^2\omega^2\beta^2}{4\pi^2 c}\left\{|C_{\parallel}(\omega)|^2 + |C_{\perp}(\omega)|^2\right\}$$

$$C_{\parallel}(\omega) = \int_{-\infty}^{\infty} dt\sin(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

$$C_{\perp}(\omega) = \int_{-\infty}^{\infty} dt\sin\theta\cos(vt/\rho)e^{i\omega(t-\frac{\rho}{c}\cos\theta\sin(vt/\rho))}$$

Synchrotron radiation geometry – using modified Bessel functions

$$K_{1/3}(\xi) = \sqrt{3}\int_0^{\infty} dx\cos\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right] \qquad K_{2/3}(\xi) = \sqrt{3}\int_0^{\infty} dx\, x\sin\left[\tfrac{3}{2}\xi\left(x+\tfrac{1}{3}x^3\right)\right]$$

Exponential factor

$$\omega\left(t_r - \frac{\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)}{c}\right) = \omega\left(t_r - \frac{\rho}{c}\cos\theta\sin(vt_r/\rho)\right)$$

In the limit of $t_r \approx 0$, $\theta \approx 0$, $v \approx c\left(1-\frac{1}{2\gamma^2}\right)$

$$\omega\left(t_r - \frac{\hat{\mathbf{r}}\cdot\mathbf{R}_q(t_r)}{c}\right) \approx \frac{\omega t_r}{2\gamma^2}\left(1+\gamma^2\theta^2\right) + \frac{\omega c^2 t_r^3}{6\rho^2} = \frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)$$

where $\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$ and $x = \frac{c\gamma t_r}{\rho\left(1+\gamma^2\theta^2\right)^{1/2}}$

Spectral form of synchrotron radiation in this case:

$$\frac{\partial^2 I}{\partial\omega\partial\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left(\frac{\omega}{\omega_c}\right)^2\left(1+\gamma^2\theta^2\right)^2\left\{\left[K_{2/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\theta^2\right)^{3/2}\right)\right]^2 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\left[K_{1/3}\left(\frac{\omega}{2\omega_c}\left(1+\gamma^2\theta^2\right)^{3/2}\right)\right]^2\right\}$$

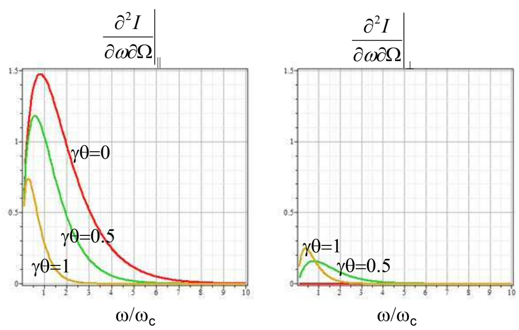
$$\omega_c \equiv \frac{3c\gamma^3}{2\rho}$$

At $\theta = 0$:

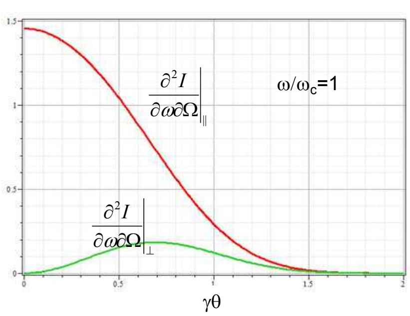
note that for $\omega \ll \omega_c \quad \Rightarrow \frac{\partial^2 I}{\partial\omega\partial\Omega} \approx \frac{q^2}{\pi^2 c}\left(\Gamma\left(\tfrac{2}{3}\right)\right)^2\left(\frac{3\omega^2\rho^2}{4c^2}\right)^{1/3}$

and for $\omega \gg \omega_c \quad \Rightarrow \frac{\partial^2 I}{\partial\omega\partial\Omega} \approx \frac{3q^2}{4\pi c}\gamma^2\left(\frac{\omega}{\omega_c}\right)^2 e^{-\omega/\omega_c}$

Plots of the intensity function for synchrotron radiation



Plots of the intensity function for synchrotron radiation



Thompson scattering – non relativistic approximation

Power radiated in direction $\hat{\mathbf{r}}$ by charged particle with acceleration $\dot{\mathbf{v}}$:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \dot{\mathbf{v}}\right)\right|^2$$

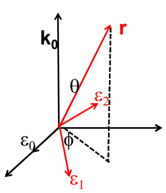
Suppose that the acceleration $\dot{\mathbf{v}}$ of a particle (charge q and mass m_q) is caused by an electric field: $\mathbf{E}(\mathbf{r},t) = \Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k}_0\cdot\mathbf{r}-i\omega t}\right)$

$$\dot{\mathbf{v}} = \frac{q}{m_q}\Re\left(\boldsymbol{\varepsilon}_0 E_0 e^{i\mathbf{k}_0\cdot\mathbf{r}-i\omega t}\right)$$

Time averaged power: $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 |E_0|^2 \left|\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\varepsilon}_0\right)\right|^2$

Thompson scattering – non relativistic approximation -- continued

Time averaged power: $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 |E_0|^2 \left|\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{\varepsilon}_0\right)\right|^2$



Polarization of incident light: $\varepsilon_0 = \hat{\mathbf{x}}$

Polarization of scattered light:

$$\varepsilon_1 = \cos\theta(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi) - \hat{\mathbf{z}}\sin\theta$$

$$\varepsilon_2 = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

Thompson scattering – non relativistic approximation -- continued

Time averaged power with polarization ε^*:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}\left(\frac{q^2}{m_q c^2}\right)^2 |E_0|^2 \left|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right|^2$$

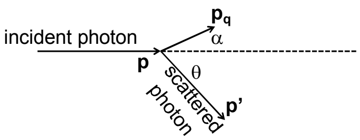
Scattered light may be polarized parallel to incident field or polarized with an angle θ so that the time and polarization averaged cross section is given by:

$$\left\langle \left|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi = \left\langle \left|\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi + \left\langle \left|\boldsymbol{\varepsilon}_2 \cdot \boldsymbol{\varepsilon}_0\right|^2 \right\rangle_\phi = \frac{1}{2}\cos^2\theta + \frac{1}{2}$$

Averaged cross section: $\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2}\right)^2 \frac{1}{2}\left(1+\cos^2\theta\right)$

This formula is appropriate in the X-ray scattering of electrons or soft γ-ray scattering of protons

Thompson scattering – relativistic and quantum modifications



Conservation of momentum and energy :

$$p = p'\cos\theta + p_q \cos\alpha \qquad pc = \hbar\omega$$

$$0 = p'\sin\theta - p_q \sin\alpha \qquad p'c = \hbar\omega'$$

$$\hbar\omega + m_q c^2 = \hbar\omega' + \sqrt{p_q{}^2 c^2 + \left(m_q c^2\right)^2}$$

$$\frac{p'}{p} = \frac{1}{1 + \frac{\hbar\omega}{m_q c^2}\left(1 - \cos\theta\right)}$$

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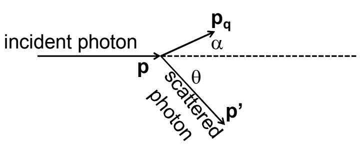
Thompson scattering – relativistic and quantum modifications



Relativistic and quantum modifications to averaged cross section :

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{q^2}{m_q c^2}\right)^2 \left(\frac{p'}{p}\right)^2 \frac{1}{2}\left(1 + \cos^2\theta\right)$$

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