Plan for Lecture 33:

Special Topics in Electrodynamics:

1. Electromagnetic aspects of superconductivity – continued
2. Review – reflection and refraction

Josephson junction -- tunneling current between two superconductors

\[ d \]

\[ B_z \]

\[ x \]
Josephson junction -- continued

\[ B(x) = \begin{cases} 
B_0 e^{i \phi_0 / 2} & x < -d / 2 \\
B_0 & -d / 2 < x < d / 2 \\
B_0 e^{-i \phi_0 / 2} & x > d / 2 
\end{cases} \]

Josephson junction -- continued

\[ A(x) = \begin{cases} 
A_0 \left( \lambda e^{i \phi_0 / 2} - (\lambda + d / 2) \right) & x < -d / 2 \\
A_0 x & -d / 2 < x < d / 2 \\
A_0 \left( -\lambda e^{-i \phi_0 / 2} + (\lambda + d / 2) \right) & x > d / 2 
\end{cases} \]

Quantum mechanical model of tunnelling current

Let \( \Psi_L = \Psi_L^0 e^{i \phi_0} \) denote a wavefunction for a Cooper pair on left

Let \( \Psi_R = \Psi_R^0 e^{i \phi_0} \) denote a wavefunction for a Cooper pair on right

\[ -i \hbar \frac{\partial \Psi_L}{\partial t} = E_L \Psi_L + \alpha \Psi_R \\
-\hbar \frac{\partial \Psi_R}{\partial t} = E_R \Psi_R + \alpha \Psi_L \]
Josephson junction -- continued

Solving for wavefunctions
\[ i \hbar \frac{\partial \Psi}{\partial t} + \frac{1}{2} \nabla^2 \Psi = - \frac{\epsilon}{\hbar} \left( \Psi \frac{\partial \Psi}{\partial n} - i \frac{\partial}{\partial n} \left( \Psi^* \frac{\partial \Psi}{\partial n} \right) \right) \]

\[ i \hbar \frac{\partial \Psi^*}{\partial t} + \frac{1}{2} \nabla^2 \Psi^* = \frac{\epsilon}{\hbar} \left( \Psi \frac{\partial \Psi^*}{\partial n} + i \frac{\partial}{\partial n} \left( \Psi^* \frac{\partial \Psi}{\partial n} \right) \right) \]

\[ \Psi^* \frac{\partial \Psi}{\partial n} = n_1 \Psi^* \frac{\partial \Psi}{\partial n} = n_1, \quad \phi_2 = \phi_1 - \phi_2 \]

\[ \frac{\partial n_1}{\partial t} = \frac{2 e}{\hbar} \sqrt{n(n_1)} \sin \phi_1 \]

\[ \frac{\partial \phi_1}{\partial t} = - \frac{E}{\hbar} - \frac{e}{\sqrt{n}} \cos \phi_1 \]

\[ \frac{\partial \phi_2}{\partial t} = \frac{E}{\hbar} - \frac{e}{\sqrt{n}} \cos \phi_2 \]

Josephson junction -- continued

Tunneling current: \( J_T = 2e \frac{\partial n_1}{\partial t} = \frac{2e}{\hbar} \sqrt{n(n_1)} \sin \phi_1 = -J_{\text{loc}} \sin \phi_{a} \)

If \( n_1 = n_{a} \) and in absence of magnetic field, \( \phi_{a}(t) = \phi_{a}(0) + \frac{E_{j} - E_{i}}{\hbar} t \)

\[ J_T = \frac{2e}{2m} \left( N \phi_1 \frac{2e}{c} + \frac{N}{2m} \phi_2 \frac{2e}{c} \right) \]

Relationship between superconductor currents \( J_L \) and \( J_R \) and tunneling current. Within the superconductor, denote the generalize current operator acting on pair wavefunction \( \Psi = \Psi^* e^{\phi} \)

\[ \dot{\Psi} = \frac{i}{2m} \left(-i \hbar N \frac{2e}{c} A \right) \]

with current \( J = \frac{2e}{2} \left( \Psi^* \frac{\partial \Psi}{\partial n} + \Psi (\frac{\partial \Psi^*}{\partial n}) \right) \)

Josephson junction -- continued

\[ \Rightarrow J_L = \frac{2e}{2m} \left( N \phi_L \frac{2e}{c} + \frac{N}{2m} \phi_1 \frac{2e}{c} \right) = 2e n_1 v_L \]

\[ \Rightarrow J_B = \frac{2e}{2m} \left( N \phi_B \frac{2e}{c} + \frac{N}{2m} \phi_2 \frac{2e}{c} \right) = 2e n_2 v_D \]

\[ \nabla \phi_L = \frac{2e}{\hbar} n_1 \frac{2e}{hc} A \quad \nabla \phi_B = \frac{2e}{\hbar} n_2 \frac{2e}{hc} A \]

Tunneling current: \( J_T = 2e \frac{\partial n_1}{\partial t} = -J_{\text{loc}} \sin \phi_2 \)

Need to evaluate \( \phi_{a} \) in presence of magnetic field
Recall that for $x \to -\infty \ v_x \to 0$ and $A \to \left( \lambda_c + d / 2 \right) B_y y$
for $x \to \infty \ v_x \to 0$ and $A \to \left( \lambda_c + d / 2 \right) B_y y$
Integrating the difference of the phase angles along $y$:
$$\phi_a = \phi_{a0} - B_y (2 \lambda_c + d) y$$
Tunneling current: $J_t = -J_{t0} \sin \phi_a$

Integrating the difference of the phase angles along $y$:
$$\phi_a = \phi_{a0} + \frac{2e}{\hbar c} B_y (2 \lambda_c + d) y$$
Tunneling current density: $J_y = J_{t0} \sin \phi_a$

Integrating current density throughout width $w$ and height $h$:
$$I_y = \frac{h}{\lambda_c} \int_{-w/2}^{w/2} \sin(\phi_{a0} + \frac{2e}{\hbar c} B_y (2 \lambda_c + d) y) \, dy$$
$$\text{where } \Phi = B_w (2 \lambda_c + d) \quad \text{and } \Phi' = \frac{2e hc}{\Phi}$$

Note: This very sensitive "SQUID" technology has been used in scanning probe techniques. See for example, J. R. Kirtley, Rep. Prog. Physics 73, 126501 (2010).
Review of reflection and refraction
Consider the normal incidence case; 3 media

\[ n_1 \quad \text{E}_0 \quad \text{E}_R \quad \text{E}_T \quad \text{E}_R \quad \text{E}_3 \quad \text{E}_T \]

Note that in this steady-state formulation, we must match the tangential components of the E and H fields at each boundary.

Each plane wave component has the form:

\[ E_i(x,t) = E_{0i} e^{j(\omega x - \beta_i t)} \]

\[ H_i(x,t) = \frac{n_i E_i}{\mu_i c} e^{j(\omega x - \beta_i t)} \]

in our case.

Matching equations:

\[ E_0 + E_3 = E_2 + E_4 \]

\[ \frac{n_1}{n_2} (E_1 - E_3) = E_1 - E_2 \]

Here:

\[ \begin{align*}
\beta_1 &= \frac{n_{1 \text{rad}}}{c} \\
E_{1\text{rad}} &= \frac{E_1}{\sin \theta_1} \\
\theta_1 &= \frac{n_{1 \text{rad}}}{c} \\
E_{1\text{rad}} &= \frac{E_1}{\sin \theta_1} \\
\end{align*} \]
Review of reflection and refraction -- continued
Consider the normal incidence case; 3 media

After some algebra:
\[ R = \left( \frac{1}{n_1} \right)^2 \left( \frac{1}{n_2} \right)^2 \left( \frac{1}{n_3} \right)^2 + 1 + \left( \frac{n_2}{n_1} \right)^2 + 2 \left( \frac{n_2}{n_1} \right)^2 \left( \frac{n_2}{n_1} \right)^2 \left( \frac{n_2}{n_1} \right)^2 \cos(2\theta) \]

Condition for zero reflectance:
\[ \cos(2\theta) = -1 \]
\[ \Rightarrow \frac{2\pi nd}{\lambda} = \frac{4\pi nd}{\lambda} \Rightarrow n_2 - (2n + 1)d \Rightarrow \frac{d}{2n} = \frac{n_2}{n_1} \]
also \( n_2 = \sqrt{n_1 n_3} \)