Electrodynamics – PHY712

Lecture 3 – Electrostatic potentials and fields

Reference: Chap. 1 in J. D. Jackson's textbook.

1. Poisson and Laplace Equations
2. Green's Theorem
3. One-dimensional examples

Poisson and Laplace Equations

We are concerned with finding solutions to the Poisson equation:

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{1}$$

and the Laplace equation:

$$\nabla^2 \Phi_L(\mathbf{r}) = 0. \tag{2}$$

In fact, the Laplace equation is the "homogeneous" version of the Poisson equation. The Green's theorem allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'. \tag{3}$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate (Dirichlet or Neumann) boundary conditions. Alternatively or in addition, boundary conditions can be adjusted using the fact that for any solution to the Poisson equation, $\Phi_P(\mathbf{r})$ other solutions may be generated by use of solutions of the Laplace equation $\Phi_P(\mathbf{r}) + C\Phi_L(\mathbf{r})$, for any constant C.

Component's of Green's Theorem

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{4}$$

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \tag{5}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'. \tag{6}$$

Example of charge density and potential varying in one dimension

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{7}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}, \tag{8}$$

with the boundary condition $\Phi(-\infty) = 0$.

Electrostatic field solution

The solution to the Poisson equation is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases} . \tag{9}$$

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0}{\varepsilon_0}(x+a) & \text{for } -a < x < 0 \\ \frac{\rho_0}{\varepsilon_0}(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} . \tag{10}$$

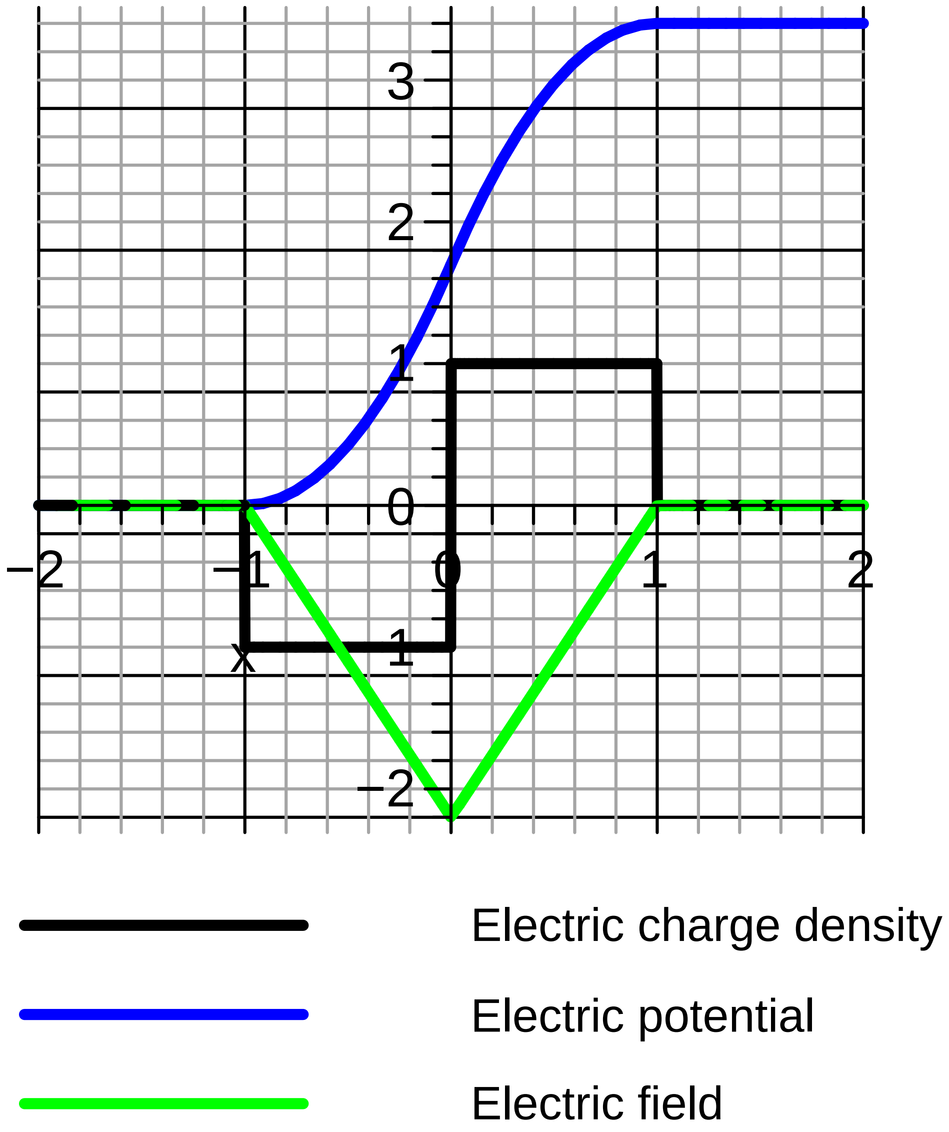
Comment about the example and solution

This particular example is one that is used to model semiconductor junctions where the charge density is controlled by introducing charged impurities near the junction. A plot of the results is given below.

The solution of the Poisson equation for this case can be determined by piecewise solution within each of the four regions. Alternatively, from Green's theorem in one-dimension, one can use the Green's function $G(x, x') = 4\pi x_<$, where,

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} G(x, x')\rho(x')dx'. \tag{11}$$

In the expression for $G(x, x')$, $x_<$ should be taken as the smaller of x and x'. It can be shown that Eq. 21 gives the identical result for $\Phi(x)$ as given in Eq. 9.


3
2
1
0
−1
−2
−2
−1
0
1
2
x
Electric charge density
Electric potential
Electric field

Notes on the one-dimensional Green's functions

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation:

$$\nabla^2 G(x, x') = -4\pi\delta(x - x'). \tag{12}$$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x. It is easily shown that with this definition of the Green's function (22), Eq. (21) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green's function which satisfies Eq. (22), we notice that we can use two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0, \tag{13}$$

where $i = 1$ or 2, to form

$$G(x, x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>). \tag{14}$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

One-dimensional Green's function – continued

In the Green's function expression appears W as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}. \tag{15}$$

We can check that this "recipe" works by noting that for $x \neq x'$, Eq. (14) satisfies the defining equation (22) by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 13. The defining equation is singular at $x = x'$, but integrating Eq. (22) over x in the neighborhood of x' $(x' - \epsilon < x < x' + \epsilon)$, gives the result:

$$\frac{dG(x,x')}{dx}\rfloor_{x=x'+\epsilon} - \frac{dG(x,x')}{dx}\rfloor_{x=x'-\epsilon} = -4\pi. \tag{16}$$

One-dimensional Green's function – continued

For example system:

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green's function is as given above. For this piecewise continuous form of the Green's function, the integration 21 can be evaluated:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}, \qquad (17)$$

which becomes

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}. \qquad (18)$$

Evaluating this expression, we find that we obtain the same result as given in Eq. (9). In general, the Green's function $G(x,x')$ solution (21) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all *neutral* charge densities, that is $\int_{-\infty}^{\infty}\rho(x)dx = 0$.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}. \tag{19}$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x'). \tag{20}$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x'). \tag{21}$$

Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x), \tag{22}$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write the Green's functions as

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}. \tag{23}$$

Example

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{24}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{25}$$

The Green's function for this case as:

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{26}$$

Example – continued

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \leq x \leq a$ from the integral

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} dx'\ G(x,x')\rho(x'), . \tag{27}$$

The boundary corrected full solution within the interval $-a \leq x \leq a$ is given by

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0} \left(16 \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2} \right) . \tag{28}$$

The above expansion apparently converges to the exact solution:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0} a^2 & \text{for } x > a \end{cases} . \tag{29}$$

Example – continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3}+\frac{1}{2}\right). \tag{30}$$