Electrodynamics – PHY712

Lecture 4 – Electrostatic potentials and fields

Reference: Chap. 1 & 2 in J. D. Jackson's textbook.

1. Mean value theorem for electrostatic potential
2. Examine Green's Theorem
3. Methods for constructing Green's functions

Future topics

1. Brief introduction to numerical methods for determining electrostatic potential
2. Method of images for planar and spherical geometries
3. Special functions associated with the electrostatic potential in various geometries

A useful theorem for electrostatics
The mean value theorem (Problem 1.10 in Jackson)

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots . \tag{1}$$

According to the premise of the theorem, we want to integrate both sides of the equation 1 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \tag{2}$$

Mean value theorem – continued

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \tag{3}$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u} \cdot \nabla = 0, \tag{4}$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^2 = \frac{4\pi R^4}{3} \nabla^2, \tag{5}$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^3 = 0, \tag{6}$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^4 = \frac{4\pi R^6}{5} \nabla^4. \tag{7}$$

Since $\nabla^2 \Phi(\mathbf{r}) = 0$, the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}), \tag{8}$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \;\; R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) \equiv \frac{1}{4\pi R^2} \int_{\text{sphere}} dS_u \Phi(\mathbf{r} + \mathbf{u}). \tag{9}$$

Since this result is independent of the radius R, we see that we have the theorem.

Review of Green's Theorem

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{10}$$

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'). \tag{11}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'. \tag{12}$$

Green's theorem – continued

For a vector field $\mathbf{A}$ in a volume V bounded by surface S, the divergence theorem states

$$\int_V d^3r \nabla \cdot \mathbf{A} = \oint_S d^2r \mathbf{A} \cdot \hat{\mathbf{r}}. \tag{13}$$

It is convenient to choose

$$\mathbf{A} = \phi \nabla \psi - \psi \nabla \phi, \tag{14}$$

where ψ and ϕ are two scalar fields.

With this choice, the divergence theorem takes the form:

$$\int_V d^3r \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) = \oint_S d^2r \left(\phi \nabla \psi - \psi \nabla \phi\right) \cdot \hat{\mathbf{r}}. \tag{15}$$

Green's theorem – continued

$$\int_V d^3r \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S d^2r \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{\mathbf{r}}. \tag{16}$$

Choose $\psi(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$ where $\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$ and $\phi(\mathbf{r}) = \Phi(\mathbf{r})$ where $\nabla^2\Phi(\mathbf{r}) = -4\pi\rho/\epsilon_0$. The divergence theorem then becomes:

$$\begin{aligned} -4\pi \int_V d^3r \left(\Phi(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r})/\epsilon_0\right) \\ = \oint_S d^2r \left(\Phi(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla\Phi(\mathbf{r})\right)\cdot\hat{\mathbf{r}}. \end{aligned} \tag{17}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'. \tag{18}$$

Methods for constructing Green's functions

For a one-dimensional system, the integral form of the Poisson equation can often be written in terms of the one-dimensional Green's function $G(x, x')$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} G(x, x')\rho(x')dx'. \tag{19}$$

There are many ways to construct the Green's function; as we will show below, a good choice is often $G(x, x') = 4\pi x_<$, meaning that in the integral $x_<$ should be taken as the smaller of x and x'.

Notes on the one-dimensional Green's functions

The Green's function for the one-dimensional Poisson equation can be defined as a solution to the equation:

$$\nabla^2 G(x, x') = -4\pi\delta(x - x'). \tag{20}$$

Here the factor of 4π is not really necessary, but ensures consistency with your text's treatment of the 3-dimensional case. The meaning of this expression is that x' is held fixed while taking the derivative with respect to x. It is easily shown that with this definition of the Green's function (20), Eq. (19) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green's function which satisfies Eq. (20), we notice that we can use two independent solutions to the homogeneous equation

$$\nabla^2 \phi_i(x) = 0, \tag{21}$$

where $i = 1$ or 2, to form

$$G(x, x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>). \tag{22}$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

One-dimensional Green's function – continued

In the Green's function expression appears W as the "Wronskian":

$$W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}. \tag{23}$$

We can check that this "recipe" works by noting that for $x \neq x'$, Eq. (22) satisfies the defining equation (20) by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 46. The defining equation is singular at $x = x'$, but integrating Eq. (20) over x in the neighborhood of x' $(x' - \epsilon < x < x' + \epsilon)$, gives the result:

$$\frac{dG(x,x')}{dx}\rfloor_{x=x'+\epsilon} - \frac{dG(x,x')}{dx}\rfloor_{x=x'-\epsilon} = -4\pi. \tag{24}$$

One-dimensional Green's function – continued

For example system:

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green's function is as given above. For this piecewise continuous form of the Green's function, the integration 29 can be evaluated:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left\{\int_{-\infty}^{x} G(x,x')\rho(x')dx' + \int_{x}^{\infty} G(x,x')\rho(x')dx'\right\}, \tag{25}$$

which becomes

$$\Phi(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty} \rho(x')dx'\right\}. \tag{26}$$

Evaluating this expression, we find that we obtain the same result as given in Eq. (**??**). In general, the Green's function $G(x,x')$ solution (29) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all *neutral* charge densities, that is $\int_{-\infty}^{\infty} \rho(x)dx = 0$.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x) u_m(x)\, dx = \delta_{nm}. \tag{27}$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \delta(x - x'). \tag{28}$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x'). \tag{29}$$

Orthogonal function expansions –continued

Therefore, if

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x), \tag{30}$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write the Green's functions as

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}. \tag{31}$$

Example

For example, consider the example discussed earlier in the interval $-a \leq x \leq a$ with

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{32}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. For this purpose, we may choose

$$u_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{[2n+1]\pi x}{2a}\right). \tag{33}$$

The Green's function for this case as:

$$G(x, x') = \frac{4\pi}{a} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right) \sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}. \tag{34}$$

Example – continued

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \leq x \leq a$ from the integral

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} dx'\; G(x, x')\rho(x'), . \tag{35}$$

The boundary corrected full solution within the interval $-a \leq x \leq a$ is given by

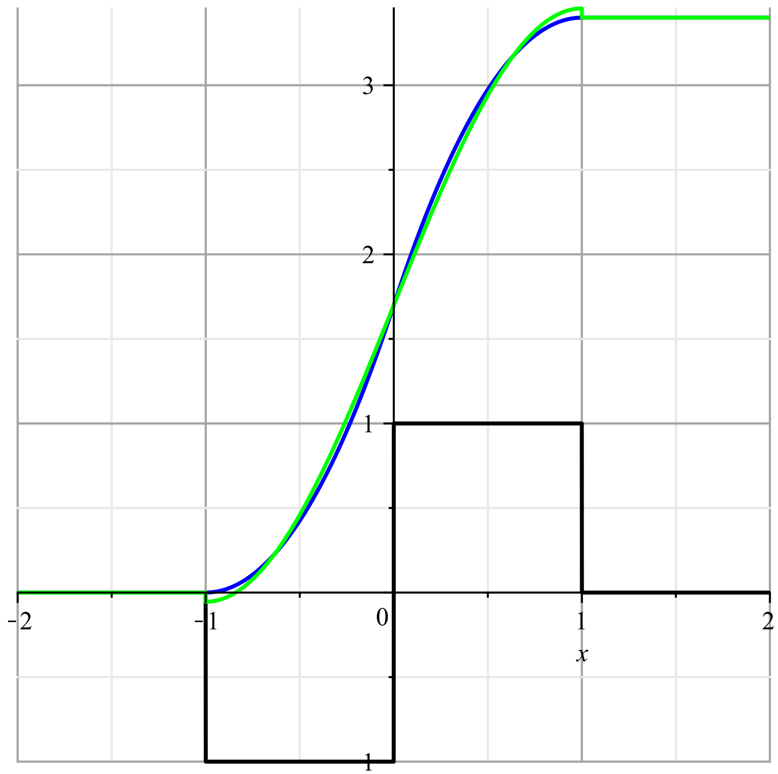
$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0} \left(16 \sum_{n=0}^{\infty} \frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2} \right). \tag{36}$$

The above expansion apparently converges to the exact solution:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0} a^2 & \text{for } x > a \end{cases} . \tag{37}$$

Example – continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3}+\frac{1}{2}\right). \tag{38}$$



Orthogonal function expansions in 2 and 3 dimensions

3 dimensions in Cartesian coordinates

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r})/\epsilon_0. \tag{39}$$

The orthogonal function expansion method can easily be extended to two and three dimensions. For example if $\{u_n(x)\}$, $\{v_n(x)\}$, and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written:

$$G(x,x',y,y',z,z') = 4\pi \sum_{lmn} \frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n}, \tag{40}$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y), \quad \text{and} \quad \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z). \tag{41}$$

See Eq. 3.167 in **Jackson** for an example.

Green's functions in 2 and 3 dimensions – continued

Combined orthogonal function expansion and homogenious solution construction

As discussed previously, an alternative method of finding Green's functions for second order ordinary differential equations is based on a product of two independent solutions of the homogeneous equation, $u_1(x)$ and $u_2(x)$, which satisfy the boundary conditions at x_1 and x_1, respectively:

$$G(x,x') = K u_1(x_<) u_2(x_>), \quad \text{where} \quad K \equiv \frac{4\pi}{\frac{du_1}{dx} u_2 - u_1 \frac{du_2}{dx}}, \tag{42}$$

with $x_<$ meaning the smaller of x and x' and $x_>$ meaning the larger of x and x'. For example, we have previously discussed the example of the one dimensional Poisson equation to have the form:

$$G(x,x') = 4\pi x_<. \tag{43}$$

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of **Jackson**. For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') g_n(y,y'). \tag{44}$$

Combined orthogonal function expansion and homogenious solution construction

The $y-$dependence of this equation will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right] g_n(y, y') = -4\pi\delta(y - y'), \tag{45}$$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right] v_{n_i}(y) = 0, \tag{46}$$

and a constant related to the Wronskian:

$$K_n \equiv \frac{4\pi}{\frac{dv_{n_1}}{dy} v_{n_2} - v_{n_1} \frac{dv_{n_2}}{dy}}. \tag{47}$$

If these functions also satisfy the appropriate boundary conditions, we can then construct the 2-dimensional Green's function from

$$G(x, x', y, y') = \sum_n u_n(x) u_n(x') K_n v_{n_1}(y_<) v_{n_2}(y_>). \tag{48}$$

Green's functions in 2 and 3 dimensions – continued

Combined orthogonal function expansion and homogenious solution construction

For example, a Green's function for a two-dimensional rectangular system with $0 \leq x \leq a$ and $0 \leq y \leq b$, which vanishes on each of the boundaries can be expanded:

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}. \tag{49}$$

As an example, we can use this result to solve the 2-dimensional Laplace equation in the square region $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary condition $\Phi(x,0) = \Phi(0,y) = \Phi(1,y) = 0$ and $\Phi(x,1) = V_0$. In this case, in determining $\Phi(x,y)$ using Eq. (18) there is no volume contribution (since the charge is zero) and the "surface" integral becomes a line integral $0 \leq x' \leq 1$ for $y' = 1$. Using the form from Eq. (49) with $a = b = 1$, it can be shown that the result takes the form:

$$\Phi(x,y) = \sum_{n=0}^{\infty} 4V_0 \frac{\sin[(2n+1)\pi x]\sinh[(2n+1)\pi y]}{(2n+1)\pi\sinh[(2n+1)\pi]} \tag{50}$$