PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 6:

Continue reading Chapter 2

1. **Methods of images -- planes, spheres**
2. **Solution of Poisson equation in for other geometries -- cylindrical**

	Lecture date	JDJ Reading	Topic	Assign.	Due date
1	Wed: 01/15/2014	Chap. 1	Introduction, units and Poisson equation	#1	01/31/2014
2	Thu: 01/16/2014	Chap. 1	Electrostatic energy calculations	#2	01/31/2014
3	Fri: 01/17/2014	Chap. 1	Poisson equation and Green's theorm	#3	01/31/2014
	Mon: 01/20/2014		MLK Holiday - no class		
4	Wed: 01/22/2014	Chap. 1	Green's functions for cartesian coordinates	#4	01/31/2014
5	Thu: 01/23/2014	Chap. 1	Brief introduction to numerical methods	#5	01/31/2014
6	Fri: 01/24/2014	Chap. 2	Method of images	#6	01/31/2014
	Mon: 01/27/2014		NAWH out of town - no class		
	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/03/2014

Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

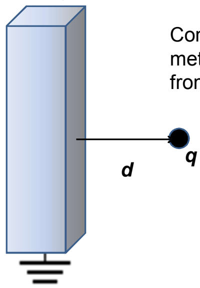
1. Direct integration of differential equation
2. Green's function techniques
3. Orthogonal function expansions
4. Method of images

Method of images

Clever trick for specialized geometries:

- Flat plane (surface)
- Sphere

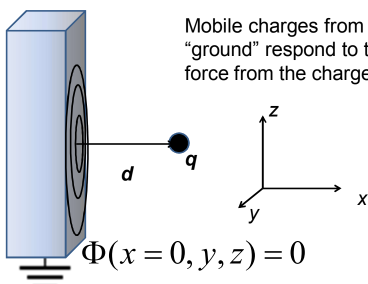
Planar case:



Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

Mobile charges from the "ground" respond to the force from the charge ***q***.



$$\Phi(x=0,y,z)=0$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{r}-d\hat{\mathbf{x}})$$

$$\Phi(x=0,y,z)=0$$

Trick for $x \geq 0$:

$$\Phi(x\geq 0,y,z)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-d\hat{\mathbf{x}}|}-\frac{q}{|\mathbf{r}+d\hat{\mathbf{x}}|}\right)$$

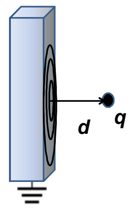
Surface charge density:

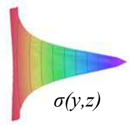
$$\sigma(y,z)=\varepsilon_0 E(0,y,z)=-\varepsilon_0\frac{d\Phi(0,y,z)}{dx}=-\frac{q}{4\pi}\left(\frac{2d}{(d^2+y^2+z^2)^{3/2}}\right)$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

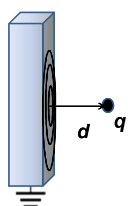
$$\text{Surface charge density}: \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note}: \iint dydz\ \sigma(y,z) = -\frac{q2d}{4\pi}2\pi\int_0^\infty \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Surface charge density:

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

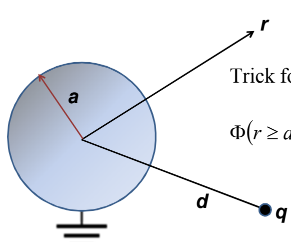
Force between charge and sheet:

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0(2d)^2}$$

Image potential between charge and sheet at distance x:

$$V(x) = \frac{-q^2}{4\pi\varepsilon_0(4x)}$$

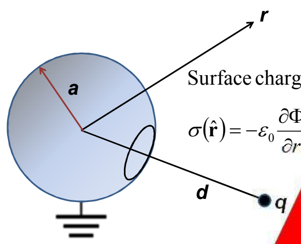
A grounded metal sphere of radius ***a***, in the presence of a point charge ***q*** at a distance ***d*** from its center.



Trick for $r \geq a$:

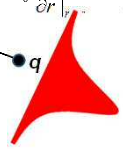
$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density :

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$



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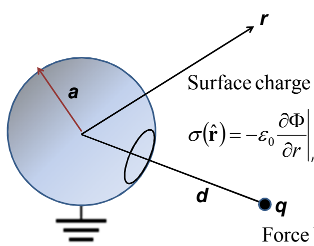


Surface charge density :

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Force between sphere and q :

$$|\mathbf{F}| = \frac{q^2}{4\pi\varepsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$

Use of image charge formalism to construct Green's function

Example :

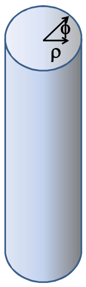
Suppose we have a Dirichlet boundary value problem on a sphere of radius a :

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a)=0$$

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \text{For} \quad r, r' > a: \qquad G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

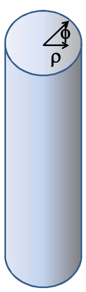
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates :

$$\Phi(\rho,\phi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\phi+\alpha_m$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):

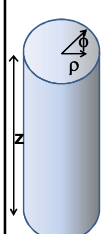


Green's function appropriate for this geometry with boundary conditions at $\rho=0$ and $\rho=\infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=\Phi(\rho,\phi+m2\pi,z)$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

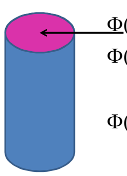
Cylindrical geometry continued:

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:

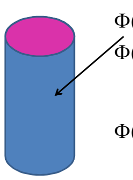


$$\Phi(\rho,\phi,z=L) = V(\rho,\phi)$$

$\Phi(\rho,\phi,z) = 0$ on all other boundaries

$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi + \alpha_{mn})$$

Cylindrical geometry example:



$$\Phi(\rho=a,\phi,z) = V(\phi,z)$$

$\Phi(\rho,\phi,z) = 0$ on all other boundaries

$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin(m\phi + \alpha_{mn})$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u), N_m(u), H_m(u)\equiv J_m(u)\pm N_m(u)$$

$$F_m^{-}(u)=I_m(u), K_m(u)$$