

**PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107**

Plan for Lecture 6:

Continue reading Chapter 2

- 1. Methods of images -- planes, spheres**
- 2. Solution of Poisson equation in for other geometries -- cylindrical**

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	Lecture date	JDJ Reading	Topic	Assign.	Due date
1	[Wed: 01/15/2014]	Chap. 1	Introduction, units and Poisson equation	#1	01/31/2014
2	[Thu: 01/16/2014]	Chap. 1	Electrostatic energy calculations	#2	01/31/2014
3	[Fri: 01/17/2014]	Chap. 1	Poisson equation and Green's theorem	#3	01/31/2014
	[Mon: 01/20/2014]		MLK Holiday - no class		
4	[Wed: 01/22/2014]	Chap. 1	Green's functions for cartesian coordinates	#4	01/31/2014
5	[Thu: 01/23/2014]	Chap. 1	Brief introduction to numerical methods	#5	01/31/2014
	[Fri: 01/24/2014]	Chap. 2	Method of images	#6	01/31/2014
	[Mon: 01/27/2014]		NAWH out of town - no class		
	[Wed: 01/29/2014]		NAWH out of town - no class		
7	[Fri: 01/31/2014]	Chap. 3	Cylindrical and spherical geometries	#7	02/03/2014

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Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

1. Direct integration of differential equation
2. Green's function techniques
3. Orthogonal function expansions
4. Method of images

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The diagram shows a vertical blue rectangular prism representing a grounded metal sheet. At the bottom center of the sheet, there are three horizontal lines indicating it is grounded. To the right of the sheet, a black sphere labeled q represents a point charge. A horizontal line extends from the sheet to the charge q , with the distance between them labeled d . The text "Consider a grounded metal sheet, a distance d from a point charge q ." is written to the right of the sheet.

A grounded metal sheet, a distance d from a point charge q .

Mobile charges from the “ground” respond to the force from the charge q .

$\Phi(x = 0, y, z) = 0$

A grounded metal sheet, a distance d from a point charge q .

$$\nabla^2 \Phi = -\frac{q}{\epsilon_0} \delta^3(\mathbf{r} - d\hat{\mathbf{x}})$$

$$\Phi(x=0, y, z) = 0$$

Trick for $x \geq 0$:

$$\Phi(x \geq 0, y, z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\mathbf{r} - d\hat{\mathbf{x}}|} - \frac{q}{|\mathbf{r} + d\hat{\mathbf{x}}|} \right)$$

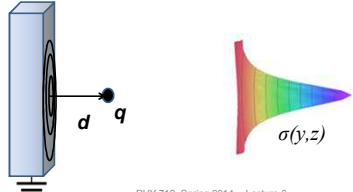
Surface charge density :

$$\sigma(y, z) = \epsilon_0 E(0, y, z) = -\epsilon_0 \frac{d\Phi(0, y, z)}{dx} = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

A grounded metal sheet, a distance d from a point charge q .

$$\text{Surface charge density : } \sigma(y,z) = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

$$\text{Note : } \iint dy dz \sigma(y,z) = -\frac{q2d}{4\pi} 2\pi \int_0^\infty \frac{udu}{(d^2 + u^2)^{3/2}} = -q$$

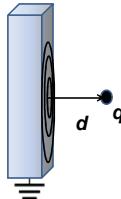


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A grounded metal sheet, a distance d from a point charge q .



Surface charge density :

$$\sigma(y,z) = -\frac{q}{4\pi} \left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}} \right)$$

Force between charge and sheet :

$$\mathbf{F} = \frac{-q^2 \hat{\mathbf{x}}}{4\pi\epsilon_0 (2d)^2}$$

Image potential between charge and sheet at distance x :

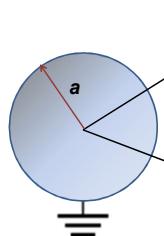
$$V(x) = \frac{-q^2}{4\pi\epsilon_0 (4x)}$$

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A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.



Trick for $r \geq a$:

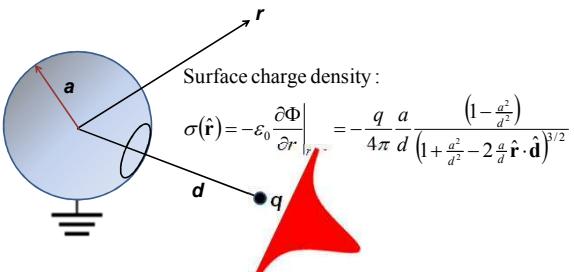
$$\Phi(r \geq a) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\mathbf{r} - \mathbf{d}|} - \frac{q}{\frac{d}{a} |\mathbf{r} - \mathbf{d} \frac{a^2}{d^2}|} \right)$$

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A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.

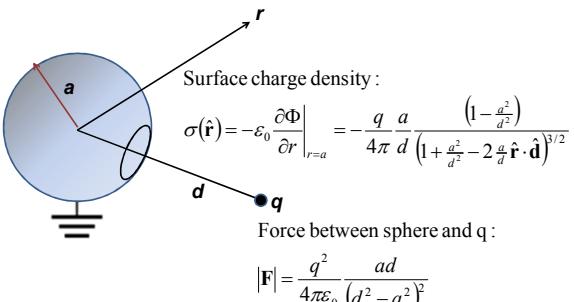


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A grounded metal sphere of radius a , in the presence of a point charge q at a distance d from its center.



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Use of image charge formalism to construct Green's function

Example :

Suppose we have a Dirichlet boundary value problem
on a sphere of radius a :

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \Phi(r = a) = 0$$

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow \text{For } r, r' > a : \quad G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\frac{a}{r'} |\mathbf{r} - \frac{a^2}{r'} \mathbf{r}'|}$$

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example): Corresponding orthogonal functions from solution of

$$\text{Laplace equation : } \nabla^2 \Phi = 0$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(\rho, \phi) = \Phi(\rho, \phi + m2\pi)$$

\Rightarrow General solution of the Laplace equation
in these coordinates :

$$\Phi(\rho, \phi) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) \sin(m\phi + \alpha_m)$$



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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no z-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) G(\rho, \rho', \phi, \phi') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(\phi - \phi')$$

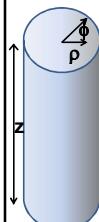
$$G(\rho, \rho', \phi, \phi') = -\ln(\rho_{>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}} \right)^m \cos(m(\phi - \phi'))$$

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



Corresponding orthogonal functions from solution of
Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = \Phi(\rho, \phi + m2\pi, z)$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

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Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

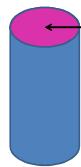
$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2} \right) R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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Cylindrical geometry example:



$$\Phi(\rho, \phi, z = L) = V(\rho, \phi)$$

$$\Phi(\rho, \phi, z) = 0 \text{ on all other boundaries}$$

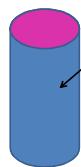
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin(m\phi + \alpha_{mn})$$

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Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \text{ on all other boundaries}$$

$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{mn})$$

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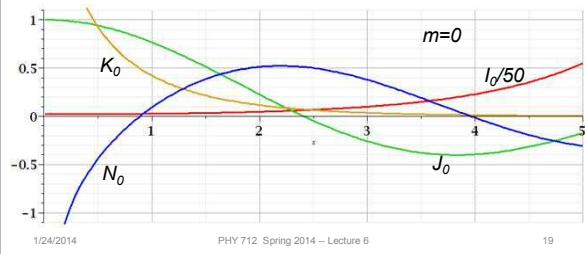
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Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^\pm(u) = 0$$

$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm N_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$



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The graph shows the evolution of normalized variables K_1 , N_1 , J_1 , and $I_1/50$ as a function of x for $m=1$. The horizontal axis (x) ranges from 0 to 5, and the vertical axis ranges from -1 to 1.
- K_1 (yellow curve) starts at approximately 1.0 and decreases rapidly towards 0.
- N_1 (blue curve) starts at 0, dips slightly below 0 around $x=0.5$, and then increases steadily.
- J_1 (red curve) starts at 0 and increases steadily.
- $I_1/50$ (green curve) starts at 0, peaks at approximately 0.6 around $x=1.5$, and then decreases back towards 0.

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