The dipole moment is defined by

$$\mathbf{p} = \int d^3r \rho(r) \mathbf{r},$$

(1)

with the corresponding potential

$$\Phi(r) = \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{p} \cdot \mathbf{\hat{r}}}{r^2},$$

(2)

and electrostatic field

$$\mathbf{E}(r) = \frac{1}{4\pi \varepsilon_0} \left\{ \frac{3\mathbf{\hat{r}}(\mathbf{p} \cdot \mathbf{\hat{r}}) - \mathbf{p}}{r^3} - \frac{4\pi}{3} \mathbf{p} \delta^3(\mathbf{r}) \right\}.$$

(3)
”Justification” of surprizing $\delta$-function term in dipole electric field

We note that Eq. (3) is poorly defined as $r \to 0$, and consider the value of a small integral of $E(r)$ about zero. (For this purpose, we are supposing that the dipole $p$ is located at $r = 0$.) In this case we will approximate

$$E(r \approx 0) \approx \left( \int_{\text{sphere}} E(r) d^3r \right) \delta^3(r). \quad (4)$$

First we note that

$$\int_{r \leq R} E(r) d^3r = -R^2 \int_{r = R} \Phi(r) \hat{r} d\Omega. \quad (5)$$
\[ \int_{r \leq R} \mathbf{E}(r) d^3r = -R^2 \int_{r=R} \Phi(r) \hat{r} d\Omega. \]  

(6)

This result follows from the Divergence theorem:

\[ \int_{\text{vol}} \nabla \cdot \mathbf{V} d^3r = \int_{\text{surface}} \mathbf{V} \cdot d\mathbf{A}. \]  

(7)

In our case, this theorem can be used to prove Eq. (11) for each cartesian coordinate if we choose \( \mathbf{V} = \hat{x} \Phi(r) \) for the \( x \)-component for example:

\[ \int_{r \leq R} \nabla \Phi(r) d^3r = \hat{x} \int_{r \leq R} \nabla \cdot (\hat{x} \Phi) d^3r + \hat{y} \int_{r \leq R} \nabla \cdot (\hat{y} \Phi) d^3r + \hat{z} \int_{r \leq R} \nabla \cdot (\hat{z} \Phi) d^3r, \]  

(8)

which is equal to

\[ \int_{r=R} \Phi(r) R^2 d\Omega ((\hat{x} \cdot \hat{r}) \hat{x} + (\hat{y} \cdot \hat{r}) \hat{y} + (\hat{z} \cdot \hat{r}) \hat{z}) = \int_{r=R} \Phi(r) R^2 d\Omega \hat{r}. \]  

(9)

Thus,

\[ \int_{r \leq R} \mathbf{E}(r) d^3r = - \int_{r \leq R} \nabla \Phi(r) d^3r = -R^2 \int_{r=R} \Phi(r) \hat{r} d\Omega. \]  

(10)
δ-function contribution dipole electric field – continued

\[ \int_{r \leq R} \mathbf{E}(r) \, d^3r = -R^2 \int_{r = R} \Phi(r) \hat{r} \, d\Omega. \] (11)

Now, we notice that the electrostatic potential can be determined from the charge density \( \rho(r) \) according to:

\[ \Phi(r) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(r')}{|r - r'|} = \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi}{2l + 1} \int d^3r' \rho(r') \frac{r'^l}{r^{l+1}} Y_{lm}^*(\hat{r}) Y_{lm}(\hat{r'}). \] (12)

We also note that the unit vector can be written in terms of spherical harmonic functions:

\[ \hat{r} = \begin{cases} 
\sin(\theta) \cos(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + \cos(\theta) \hat{z} \\
\sqrt{\frac{4\pi}{3}} \left( Y_{1-1}(\hat{r}) \frac{\hat{x} + i\hat{y}}{\sqrt{2}} + Y_{11}(\hat{r}) \frac{-\hat{x} + i\hat{y}}{\sqrt{2}} + Y_{10}(\hat{r}) \hat{z} \right) 
\end{cases} \]
\[ -R^2 \int_{r=R} \Phi(r) \hat{r} \, d\Omega = - \frac{1}{4\pi \epsilon_0} \frac{4\pi R^2}{3} \int d^3 r' \frac{\rho(r')}{r'^2} \hat{r}'. \] (13)

The choice of \( r_\prec \) and \( r_\succ \) is a choice between the integration variable \( r' \) and the sphere radius \( R \). If the sphere encloses the charge distribution \( \rho(r') \), then \( r_\prec = r' \) and \( r_\succ = R \) so that Eq. (13) becomes

\[ -R^2 \int_{r=R} \Phi(r) \hat{r} \, d\Omega = - \frac{1}{4\pi \epsilon_0} \frac{4\pi R^2}{3} \frac{1}{R^2} \int d^3 r' \rho(r') r' \hat{r}' \equiv - \frac{P}{3\epsilon_0}. \] (14)

If the charge distribution \( \rho(r') \) lies outside of the sphere, then \( r_\succ = r' \) and \( r_\prec = R \) so that Eq. (13) becomes

\[ -R^2 \int_{r=R} \Phi(r) \hat{r} \, d\Omega = - \frac{1}{4\pi \epsilon_0} \frac{4\pi R^2}{3} R \int d^3 r' \frac{\rho(r')}{r'^2} \hat{r'} \equiv \frac{4\pi R^3}{3} E(0). \] (15)