PHY 770 -- Statistical Mechanics
12:00* - 1:45 PM TR Olin 107

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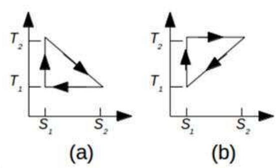
Lecture 17

Chap. 7 – Brownian motion and other non-equilibrium phenomena
- **Comments on mid-term exam**
- **Fokker-Planck equation**
- **Diffusion solutions**

***Partial make-up lecture -- early start time**

2	Thu: 01/16/2014	Chap. 3	Review of macroscopic thermodynamics	#2	02/04/2014
3	Tue: 01/21/2014	Chap. 3	Thermodynamic potentials	#3	02/04/2014
4	Tue: 01/21/2014	Chap. 3	Thermodynamic stability	#4	02/04/2014
5	Thu: 01/23/2014	Chap. 3	Thermodynamic stability	#5	02/04/2014
	Tue: 01/28/2014		NAWH out of town - no class		
	Thu: 01/30/2014		NAWH out of town - no class		
6	Tue: 02/04/2014	Chap. 4	Phase transitions	#6	02/11/2014
7	Thur: 02/06/2014	Chap. 2	Microscopic analysis of entropy	#7	02/11/2014
8	Thur: 02/06/2014	Chap. 2	Microscopic analysis of entropy	#8	02/11/2014
9	Tue: 02/11/2014	Chap. 2	Microscopic analysis of entropy	#9	02/18/2014
	Thur: 02/13/2014		Class cancelled due to weather		
10	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#10	02/27/2014
11	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#11	02/27/2014
12	Thur: 02/20/2014	Chap. 5	The Ising model (class 12-1:45 PM)	#12	02/27/2014
	Tue: 02/25/2014		NAWH out of town -- no class		
13	Thur: 02/27/2014	Chap. 6	Grand partition function (class 12-1:45 PM)		
	Tue: 03/04/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Thur: 03/06/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Tue: 03/11/2014		Spring break (no class meeting)		
	Thur: 03/13/2014		Spring break (no class meeting)		
14	Tue: 03/18/2014	Chap. 6	Fermi and Bose particles (class 12-1:45 PM; Exam due)	#13	03/25/2014
15	Thur: 03/20/2014	Chap. 6	Interacting particles (class 12-1:45 PM)	#14	03/25/2014
16	Tue: 03/25/2014	Chap. 7	Langevin equation (class 12-1:45 PM)	#15	04/01/2014
17	Thur: 03/27/2014	Chap. 7	Fokker-Planck equation (class 12-1:45 PM)	#16	04/01/2014

Comments on the mid-term exam



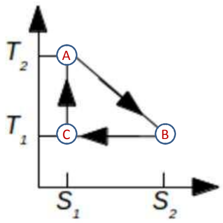
1.

The figure above shows the temperature-entropy diagrams for two reversible heat engines operating between temperatures T_1 and T_2 and entropies S_1 and S_2. The answers to the following questions can be expressed in terms of these 4 parameters.

(a) For process (a), find the net heat and work (W_{net}), the heat input (Q_{in}), and the efficiency ($\eta \equiv W_{net}/Q_{in}$).

(b) For process (b), find the net heat and work (W_{net}), the heat input (Q_{in}), and the efficiency ($\eta \equiv W_{net}/Q_{in}$).

(c) Compare the efficiencies of (a) and (b) with each other and with a Carnot cycle of this system operating between the same temperatures and entropies.



(a)

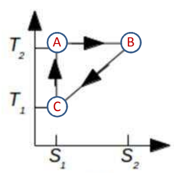
$$Q_{A\to B}=\int_{S_1}^{S_2} TdS=\frac{1}{2}\left(T_2+T_1\right)\left(S_2-S_1\right)$$

$$Q_{B\to C}=\int_{S_2}^{S_1} TdS=-T_1\left(S_2-S_1\right)$$

$$Q_{C\to A}=0$$

$$W_{net}=Q_{A\to B}+Q_{B\to C}+Q_{C\to A}=\frac{\left(S_2-S_1\right)\left(T_2-T_1\right)}{2}$$

$$\eta=\frac{W_{net}}{Q_{A\to B}}=\frac{\left(S_2-S_1\right)\left(T_2-T_1\right)}{\left(S_2-S_1\right)\left(T_2+T_1\right)}=\frac{\left(T_2-T_1\right)}{\left(T_2+T_1\right)}$$



(b)

$$Q_{A\to B}=\int_{S_1}^{S_2} TdS=T_2\left(S_2-S_1\right)$$

$$Q_{B\to C}=\int_{S_2}^{S_1} TdS=-\frac{1}{2}\left(T_2+T_1\right)\left(S_2-S_1\right)$$

$$Q_{C\to A}=0$$

$$W_{net}=Q_{A\to B}+Q_{B\to C}+Q_{C\to A}=\frac{\left(S_2-S_1\right)\left(T_2-T_1\right)}{2}$$

$$\eta=\frac{W_{net}}{Q_{A\to B}}=\frac{\left(S_2-S_1\right)\left(T_2-T_1\right)}{2\left(S_2-S_1\right)T_2}=\frac{\left(T_2-T_1\right)}{2T_2}$$

For Carnot cycle: $\eta=\frac{\left(T_2-T_1\right)}{T_2}$

$$\eta_{Carnot}=2\eta_b=\left(\frac{T_2+T_1}{T_2}\right)\eta_a$$

2. Consider a system of N non-interacting particles each with a given total angular momentum quantum number $J = 1/2, 1, 3/2, 2...$ with $2J+1$ states $|J, M\rangle$ with $M = -J, -J+1...J-1, J$. In an applied magnetic field B, the expection values of the Hamiltonian is given by

$$\langle J, M|\mathcal{H}|J, M\rangle = -\mu M B,$$

where μ is the magnetic moment constant for the particles.

(a) Evaluate the canonical partition function for this system by summing the geometric series.

(b) Write an expression for the Helmholz free energy of the system.

(c) For fixed temperature, evaluate the form of the magnetization and the magnetic susceptibility.

(d) For fixed magnetic field, evaluate the form of the heat capacity.

(e) For $J = 2$ and convenient choices for the other parameters of the system, plot the heat capacities and magnetic functions.

Canonical partition function:

$$Z_N(T) = \left(\mathrm{Tr}[e^{-\beta H}]\right)^N = \left(\sum_{M=-J}^{J} e^{\beta\mu MB}\right)^N$$

Using formula for summation of geometric series:

$$\sum_{M=-J}^{J} e^{\beta\mu MB} = \frac{e^{-\beta\mu JB} - e^{\beta\mu(J+1)B}}{1-e^{\beta\mu B}} = \frac{\sinh\left(\beta\mu(J+1/2)B\right)}{\sinh\left(\beta\mu B/2\right)}$$

Helmholz free energy:

$$A(T,B) = -kT\ln\left(Z_N(T)\right) = -kTN\left(\ln\left(\frac{\sinh\left(\beta\mu(J+1/2)B\right)}{\sinh\left(\beta\mu B/2\right)}\right)\right)$$

Magnetization:

$$\mathcal{M} = -\frac{\partial A(T,B)}{\partial B} = \mu N\left((J+1/2)\coth\left(\beta\mu(J+1/2)B\right) - (1/2)\coth\left(\beta\mu B/2\right)\right)$$

Magnetic susceptibility:

$$\chi = \frac{\partial \mathcal{M}}{\partial B} = \frac{\mu^2 N}{kT}\left(\frac{(J+1/2)^2}{\sinh^2\left(\beta\mu(J+1/2)B\right)} - \frac{(1/2)^2}{\sinh^2\left(\beta\mu B/2\right)}\right)$$

Internal energy:

$$U(T,B) = -\frac{\partial \ln\left(Z_N(T,B)\right)}{\partial\beta} = -N\mu B\left((J+1/2)\coth\left(\beta\mu(J+1/2)B\right) - (1/2)\coth\left(\beta\mu B/2\right)\right)$$

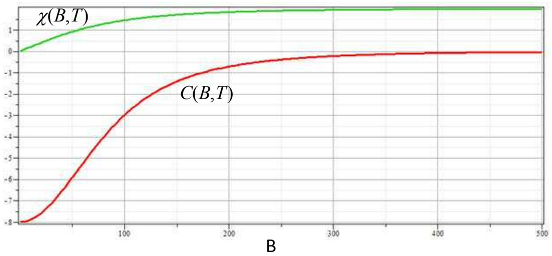
Heat capacity:

$$C(T,B) = \frac{\partial U(T,B)}{\partial T} = \frac{N\mu^2 B^2}{kT^2}\left(\frac{(J+1/2)^2}{\sinh^2\left(\beta\mu(J+1/2)B\right)} - \frac{(1/2)^2}{\sinh^2\left(\beta\mu B/2\right)}\right)$$

Evaluation for $J = 2$:

$$C(T,B) = \frac{N\mu^2 B^2}{4kT^2}\left(\frac{25}{\sinh^2\left(\beta\mu(5/2)B\right)} - \frac{1}{\sinh^2\left(\beta\mu B/2\right)}\right)$$

$$\mathcal{M} = \mu N\left((5/2)\coth\left(\beta\mu(5/2)B\right) - (1/2)\coth\left(\beta\mu B/2\right)\right)$$



3. Consider a system of N independent harmonic oscillators each with a mass m and frequency ω.
   (a) Suppose the oscillators are quantum mechanical, each having the discrete energy levels
   $$\epsilon_n = \hbar\omega\left(n + \frac{1}{2}\right),$$
   where n takes integer values $n = 0, 1, 2, \ldots\infty$. Find the canonical partition function for this case and the corresponding Helmholz free energy, entropy, internal energy, and heat capacity.
   (b) Suppose that the oscillators are classical oscillators, each have the continuous energies
   $$\epsilon(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2},$$
   where q and p are the one-dimensional coordinate and momentum variables of the phase space of each oscillator. Find the canonical partition function for this case and the corresponding Helmholz free energy, entropy, internal energy, and heat capacity for this case, and compare with the quantum case.

Quantum treatment:

Partition function: $Z_N(T) = \left(\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}\right)^N$

Helmholz free energy: $A = -kT\ln\left(Z_N(T)\right) = \frac{N\hbar\omega}{2} + NkT\ln\left(1-e^{-\beta\hbar\omega}\right)$

Entropy: $S = \frac{\partial A}{\partial T} = Nk\ln\left(1-e^{-\beta\hbar\omega}\right) + \frac{N\hbar\omega}{T}\left(\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\right)$

Internal energy: $U = -\frac{\partial\ln\left(Z_N(T)\right)}{\partial\beta} = \frac{N\hbar\omega}{2} + N\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}\right)$

Heat capacity: $C = \frac{\partial U}{\partial T} = \frac{N\hbar^2\omega^2}{kT^2}\left(\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}\right)$

Classical treatment:

Partition function:

$$Z_N(T) = \frac{1}{N!}\left(\frac{L}{2\pi\hbar}\int dp\, e^{-\beta p^2/(2m)} \frac{1}{L}\int dx\, e^{-\beta m\omega^2 x^2/2}\right)^N$$
$$= \frac{1}{N!}\left(\frac{kT}{\hbar\omega}\right)^N$$

Classical internal energy: $U = NkT$

Classical specific heat: $C = Nk$

The high temperature limit of the quantum result:

$$\frac{N\hbar^2\omega^2}{kT^2}\left(\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}\right) \approx \frac{N\hbar^2\omega^2}{kT^2}\left(\frac{1}{\left(\beta\hbar\omega\right)^2}\right) = Nk$$

4. Consider a system of non-interacting Fermi particles of mass m moving in 3-dimensions within a volume V. For this system, assume that the degeneracy factor is $g = 2$.

(a) Write the form of the grand partition function $Z_{FD}(T, V, \mu)$ as a function of T, V, and μ (the chemical potential). You may wish to follow the derivation of your textbook, but supply all intermediate steps.

(b) Derive expressions for the grand potential $\Omega_{FD}(T, V, \mu)$, the average particle density $\langle N\rangle/\langle V\rangle$, and the pressure P. Express your answers in terms of the functions $f_{5/2}(z)$ and $f_{3/2}(z)$.

(c) Now consider the high temperature limit. Derive an expression for the average particle density $\langle N\rangle/\langle V\rangle$ keeping two terms to leading order in $1/T$.

(d) Now consider the low temperature limit. Derive an expression for the average particle density $\langle N\rangle/\langle V\rangle$ keeping two terms to leading order in T.

(e) How can you determine the chemical potential at arbitrary temperature for this system?

Partition function: $Z_{FD}(T) = \prod_p \left(1 + e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\right)^2$

$$\Omega = -kT\ln\left(Z_{FD}(T)\right) = -2kT\sum_p \ln\left(1 + e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\right)$$

$$\langle N\rangle = -\frac{\partial\Omega}{\partial\mu} = 2\sum_p\left(\frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}+1}\right)$$

$$\sum_p \rightarrow \frac{V}{(2\pi\hbar)^3}4\pi\int_0^\infty p^2 dp \rightarrow 4\pi V\left(\frac{m}{2\pi^2\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{\varepsilon}\, d\varepsilon$$

$$\frac{\langle N\rangle}{V}=8\pi\left(\frac{m}{2\pi^2\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{\varepsilon}\, d\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

For $T\to 0,\quad \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\approx\begin{cases}1 & \text{for } \varepsilon<\mu(0)\\ 0 & \text{for } \varepsilon<\mu(0)\end{cases}$

$$\frac{\langle N\rangle}{V}\approx 8\pi\left(\frac{m}{2\pi^2\hbar^2}\right)^{3/2}\int_0^\mu \sqrt{\varepsilon}\, d\varepsilon=\frac{16\pi}{3}\left(\frac{m\mu(0)}{2\pi^2\hbar^2}\right)^{3/2}$$

$$\mu(0)=\frac{\hbar^2}{2m}\left(3\pi^2\frac{\langle N\rangle}{V}\right)^{3/2}$$

Alternatively, following your textbook: Define $z\equiv e^{\beta\mu}\equiv e^{\nu}$

From Eq. 6.97 & 6.101: $\frac{\langle N\rangle}{V}=\frac{1}{\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2}\int_{-\nu}^\infty dt\frac{e^t}{\left(e^t+1\right)^2}\left(\frac{\nu^{3/2}}{3}+\frac{1}{2}\nu^{1/2}t+\ldots\right)$

$$\approx\frac{1}{3\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2}\left(\left(\frac{\mu}{kT}\right)^{3/2}+\frac{\pi^2}{8}\left(\frac{\mu}{kT}\right)^{-1/2}+\ldots\right)$$

$$=\frac{1}{3\pi^2}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\left(1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^{2}+\ldots\right)$$

$$\mu(T)\approx\mu(0)\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\mu(0)}\right)^2+\ldots,\right)$$

$$\frac{\langle N\rangle}{V}=8\pi\left(\frac{m}{2\pi^2\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{\varepsilon}\, d\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

General treatment: $z=e^{\beta\mu}$

$$\frac{\langle N\rangle}{V}=2\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} f_{3/2}(z)=2\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\frac{4}{\sqrt{\pi}}\int_0^\infty x^2dx\frac{z}{e^{x^2}+z}$$

For general temperature, $f_{3/2}(z)$ can be evaluated numerically to find z which matches

$$f_{3/2}(z)=\frac{\langle N\rangle}{2V}\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}$$

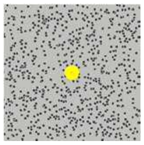
For high temperature $z\to 0$ and $f_{3/2}(z)$ can be approximated with power series:

$$\frac{\langle N\rangle}{V}=2\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\left(z-\frac{z^2}{2^{3/2}}+\frac{z^3}{3^{3/2}}-\ldots\right)$$

Leading term: $\mu=kT\ln(z)\approx kT\ln\left(\frac{\langle N\rangle}{2V}\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}\left(1-\frac{\langle N\rangle}{2^{5/2}V}\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}-\ldots\right)\right)$

Brownian motion:

Phonomenon: Under a microscope a large particle (~1 μ in diameter) immersed in a fluid with the same density as the particle, appears to be in a state of agitation, undergoing rapid and random motions.



http://upload.wikimedia.org/wikipedia/commons/c/c2/Brownian_motion_large.gif

Probability analysis of Brownian motion -- Fokker-Planck equation

Rather than analyzing single Brownian particles, it is convenient to analyze their probability density

$\rho(x,v,\xi,t)$: probability of finding the Brownian particle at time t with
position between x and $x+dx$
velocity between v and $v+dv$
random force ξ

Averaging over random forces: $P(x,v,t)=\langle \rho(x,v,\xi,t)\rangle_\xi$

Fokker-Planck equation:

g comes from random force: $\langle \xi(t_1)\xi(t_2)\rangle_\xi = g\delta(t_1-t_2)$

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m}v - \frac{1}{m}F(x)\right)P\right) + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2}$$

$F(x) = -\frac{\partial V}{\partial x}$ Force on particle due to potential $V(x)$

Probability analysis of Brownian motion Fokker-Planck equation

Justification of Fokker-Planck equation

Langevin equation in presence of friction (γ) and potential force ($F(x) = -\nabla V(x)$):

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}F(x) + \frac{1}{m}\xi(t) \qquad v(t) = \frac{dx(t)}{dt}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(v\rho)}{\partial x} + \frac{\gamma}{m}\frac{\partial(v\rho)}{\partial v} - \frac{1}{m}F(x)\frac{\partial \rho}{\partial v} - \frac{1}{m}\xi(t)\frac{\partial \rho}{\partial v}$$

Averaging over the stochastic force:

$$\frac{\partial\langle\rho(t)\rangle_\xi}{\partial t} = -\left(v\frac{\partial}{\partial x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial v} + \frac{1}{m}F(x)\frac{\partial}{\partial v}\right)\langle\rho(t)\rangle_\xi + \frac{g}{2m^2}\frac{\partial^2\langle\rho(t)\rangle_\xi}{\partial v^2}$$

Recall: $P(x,v,t)=\langle \rho(x,v,\xi,t)\rangle_\xi$

Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m}v - \frac{1}{m}F(x)\right)P\right) + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2}$$

Probability analysis of Brownian motion Fokker-Plank equation

Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m}v - \frac{1}{m}F(x)\right)P\right) + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2}$$

Define probability current:

$$\mathbf{J} = \hat{\mathbf{x}}vP - \hat{\mathbf{v}}\left(\frac{\gamma}{m}vP - \frac{1}{m}F(x)P + \frac{g}{2m^2}\frac{\partial P}{\partial v}\right)$$

$$\frac{\partial P}{\partial t} = -\nabla\cdot\mathbf{J} \qquad \text{where } \nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{v}}\frac{\partial}{\partial v}$$

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Example solution in the limit of large friction

Langevin equation in presence of friction (γ) and potential force ($F(x) = -\nabla V(x)$):

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}F(x) + \frac{1}{m}\xi(t) \qquad v(t) = \frac{dx(t)}{dt}$$

When γ is sufficiently large, the system reaches a steady-state very rapidly so the $\frac{dv(t)}{dt} \approx 0$ Then the Langevin equation reduces to: $\frac{dx(t)}{dt} = \frac{1}{\gamma}F(x) + \frac{1}{\gamma}\xi(t)$

The Fokker-Planck equation reduces to:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(\left(\frac{1}{\gamma}F(x)\right)P\right) + \frac{g}{2\gamma^2}\frac{\partial^2 P}{\partial x^2}$$

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Example solution in the limit of large friction -- continued

In this case:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(\left(\frac{1}{\gamma}F(x)\right)P\right) + \frac{g}{2\gamma^2}\frac{\partial^2 P}{\partial x^2}$$

Further consider the case where $F(x) = 0$:

$$\frac{\partial P}{\partial t} = \frac{g}{2\gamma^2}\frac{\partial^2 P}{\partial x^2} \equiv D\frac{\partial^2 P}{\partial x^2} \qquad \text{Note: } D = \frac{g}{2\gamma^2} = \frac{2\gamma kT}{2\gamma^2} = \frac{kT}{\gamma}$$

Solution of this diffusion equation:

Let $P(x,t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dq e^{-iqx}\tilde{P}(q,t)$

$$\frac{\partial \tilde{P}(q,t)}{\partial t} = -q^2 D\tilde{P}(q,t) \qquad \Rightarrow \tilde{P}(q,t) = Ce^{-Dq^2 t}$$

$$P(x,t) \equiv \frac{C}{2\pi}\int_{-\infty}^{\infty} dq e^{-iqx}e^{-Dq^2 t} = \frac{C}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

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Example solution in the limit of large friction -- continued

In this limit with the force $F(x) = -\frac{dV(x)}{dx}$

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\left(\frac{1}{\gamma}\frac{dV(x)}{dx}\right)P\right) + \frac{g}{2\gamma^2}\frac{\partial^2 P}{\partial x^2}$$

$$\frac{\partial P}{\partial t} = \frac{1}{\gamma}\frac{d^2V(x)}{dx^2}P + \frac{1}{\gamma}\frac{dV(x)}{dx}\frac{\partial P}{\partial x} + \frac{g}{2\gamma^2}\frac{\partial^2 P}{\partial x^2}$$

Using the scaled time $t = \gamma\tau$

$$\frac{\partial P}{\partial \tau} = \frac{d^2V(x)}{dx^2}P + \frac{dV(x)}{dx}\frac{\partial P}{\partial x} + \frac{g}{2\gamma}\frac{\partial^2 P}{\partial x^2}$$

Example solution in the limit of large friction -- continued

$$\frac{\partial P}{\partial \tau} = \frac{d^2V(x)}{dx^2}P + \frac{dV(x)}{dx}\frac{\partial P}{\partial x} + \frac{g}{2\gamma}\frac{\partial^2 P}{\partial x^2}$$

Convenient transformation:

$$P(x,\tau) \equiv e^{-\frac{\gamma V(x)}{g}}\Psi(x,\tau)$$

$$\frac{\partial \Psi(x,\tau)}{\partial \tau} = \left(\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{\gamma}{2g}\left(\frac{dV(x)}{dx}\right)^2\right)\Psi(x,\tau) + \frac{g}{2\gamma}\frac{\partial^2 \Psi(x,\tau)}{\partial x^2}$$

$$\equiv -H_{FP}\Psi(x,\tau)$$

Find eigenvalues and eigenfunctions of H_{FP} :

$$H_{FP}(x) \equiv -\left(\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{\gamma}{2g}\left(\frac{dV(x)}{dx}\right)^2\right) - \frac{g}{2\gamma}\frac{\partial^2}{\partial x^2}$$

$H_{FP}(x)\phi_n(x) = \lambda_n\phi_n(x)$ Note, since $H_{FP}(x)$ is self-adjoint, $\{\phi_n(x)\}$ form complete orthonormal set

Example solution in the limit of large friction -- continued

$$\frac{\partial \Psi(x,\tau)}{\partial \tau} = \left(\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{\gamma}{2g}\left(\frac{dV(x)}{dx}\right)^2\right)\Psi(x,\tau) + \frac{g}{2\gamma}\frac{\partial^2 \Psi(x,\tau)}{\partial x^2}$$

Expansion in terms of eigenfunctions of $H_{FP}(x)$:

$$\Psi(x,\tau) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \tau}\phi_n(x)$$

Evaluation of eigenfunctions:

Note that $\lambda_0 = 0$ for:

$$\left(\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{\gamma}{2g}\left(\frac{dV(x)}{dx}\right)^2\right)\phi_0(x) + \frac{g}{2\gamma}\frac{\partial^2 \phi_0(x)}{\partial x^2} = 0$$

$$\Rightarrow \phi_0(x) = Ce^{-\frac{\gamma}{g}V(x)}$$

$$P(x,\tau) = e^{-\frac{\gamma}{g}V(x)}\Psi(x,\tau) = \left(\phi_0(x)\right)^2 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n \tau}\phi_0(x)\phi_n(x)$$

Example solution in the limit of large friction -- continued

Conservation of probabilty condition:

$$\int_{-\infty}^{\infty} dx P(x,\tau) = 1 \quad \Rightarrow \int_{-\infty}^{\infty} dx \left(\phi_0(x)\right)^2 = 1$$

Determining unknown coeffients a_n : Note: $\int_{-\infty}^{\infty} dx \phi_n(x)\phi_{n'}(x) = \delta_{nn'}$

$$P(x,0) = \left(\phi_0(x)\right)^2 + \sum_{n=1}^{\infty} a_n \phi_0(x)\phi_n(x)$$

$$\Rightarrow a_n = \int_{-\infty}^{\infty} dx \phi_n(x) \left(\frac{P(x,0)}{\phi_0(x)}\right)$$

Also note that if we can prove that $\lambda_n > 0$ for $n \geq 1$, then:

$$P(x, \tau \to \infty) = \left(\phi_0(x)\right)^2$$

Example solution in the limit of large friction -- continued

Example -- let $V(x) = \frac{1}{2}\kappa x^2$:

$$H_{FP}(x) \equiv -\left(\frac{1}{2}\frac{d^2V(x)}{dx^2} - \frac{\gamma}{2g}\left(\frac{dV(x)}{dx}\right)^2\right) - \frac{g}{2\gamma}\frac{d^2}{dx^2}$$

$$= -\left(\frac{\kappa}{2} - \frac{\gamma\kappa^2}{2g}x^2\right) - \frac{g}{2\gamma}\frac{d^2}{dx^2}$$

$$= \frac{g}{2\gamma}\left(-\frac{d^2}{dx^2} + \left(\frac{\gamma\kappa}{g}\right)^2 x^2 - \frac{\gamma\kappa}{g}\right)$$

$$= \frac{\kappa}{2}\left(-\left(\frac{g}{\gamma\kappa}\right)\frac{d^2}{dx^2} + \left(\frac{\gamma\kappa}{g}\right)x^2 - 1\right)$$

$$= \frac{\kappa}{2}\left(-\frac{d^2}{dy^2} + y^2 - 1\right) \qquad \text{for } y = \sqrt{\frac{\gamma\kappa}{g}}x$$

Example solution in the limit of large friction -- continued

Finding eigenvalues and eigenfunctions:

$$H_{FP}(x) = \frac{\kappa}{2}\left(-\frac{d^2}{dy^2} + y^2 - 1\right) \qquad \text{for } y = \sqrt{\frac{\gamma\kappa}{g}}x$$

Note the example of the Harmonic Oscillator in quantum mechanics:

$$\left(-\frac{d^2}{dy^2} + y^2\right) f_n(y) = \nu_n f_n(y)$$

$f_n(y) = e^{-y^2/2} H_n(y)$ for $H_n(y) \equiv$ Hermite polynomials

$\nu_n = 2n + 1$ for $n = 0, 1, \ldots\ldots$

$$\Rightarrow H_{FP}(x)\phi_n(x) = \lambda_n \phi_n(x) \qquad \lambda_n = \kappa n$$

$$\phi_n(x) = C_n e^{-\frac{\gamma\kappa}{2g}x^2} H_n\left(\sqrt{\frac{\gamma\kappa}{g}}x\right)$$