PHY 770 -- Statistical Mechanics
12:00* - 1:45 PM TR Olin 107

Instructor: Natalie Holzwarth (Olin 300)
Course Webpage: http://www.wfu.edu/~natalie/s14phy770

Lecture 19

Chap. 9 – Transport coefficients
- ❑ **"Elementary" transport theory**
- ❑ **The Boltzmann equation**

***Partial make-up lecture -- early start time**

8	Thur: 02/06/2014	Chap. 2	Microscopic analysis of entropy	#8	02/11/2014
9	Tue: 02/11/2014	Chap. 2	Microscopic analysis of entropy	#9	02/18/2014
	Thur: 02/13/2014		Class cancelled due to weather		
10	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#10	02/27/2014
11	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#11	02/27/2014
12	Thur: 02/20/2014	Chap. 5	The Ising model (class 12-1:45 PM)	#12	02/27/2014
	Tue: 02/25/2014		NAWH out of town -- no class		
13	Thur: 02/27/2014	Chap. 6	Grand partition function (class 12-1:45 PM)		
	Tue: 03/04/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Thur: 03/06/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Tue: 03/11/2014		Spring break (no class meeting)		
	Thur: 03/13/2014		Spring break (no class meeting)		
14	Tue: 03/18/2014	Chap. 6	Fermi and Bose particles (class 12-1:45 PM; Exam due)	#13	03/25/2014
15	Thur: 03/20/2014	Chap. 6	Interacting particles (class 12-1:45 PM)	#14	03/25/2014
16	Tue: 03/25/2014	Chap. 7	Langevin equation (class 12-1:45 PM)	#15	04/01/2014
17	Thur: 03/27/2014	Chap. 7	Fokker-Planck equation (class 12-1:45 PM)	#16	04/03/2014
18	Tue: 04/01/2014	Chap. 7	Linear Response (class 12-1:45 PM)	#17	04/10/2014
19	Thur: 04/03/2014	Chap. 9	Transport theory (class 12-1:45 PM)	#18	4/10/2014
20	Tue: 04/08/2014		(class 12-1:45 PM)		
21	Thur: 04/10/2014		(class 12-1:45 PM)		
22	Tue: 04/15/2014		(class 12-1:45 PM)		
23	Thur: 04/17/2014		(class 12-1:45 PM)		
24	Tue: 04/22/2014		(class 12-1:45 PM)		
	Thur: 04/24/2014		Presentations Part I (class 11-1:45 PM)		
	Tue: 04/29/2014		Presentations Part II (class 11-1:45 PM)		

What is transport theory?

Mathematical description of the averaged motion of particles or other variables through a host medium.

Examples of transport parameters
- Thermal conductivity
- Electrical conductivity
- Diffusion coefficients
- Viscosity

Simple transport theory

Base system – low density gas near thermal equilibrium; assume that the interaction energy is negligible compared to the kinetic energy of the particles.

For N particles in volume V, the probability of finding a particle with velocity between v and $v+dv$:

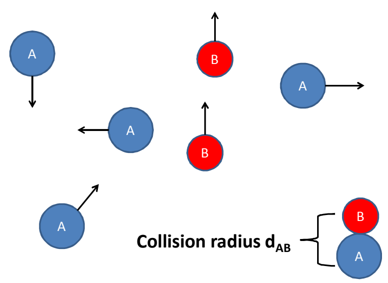
$$F(v)=\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}$$

Define the mean free path λ as the average distance a particle travels between collisions. Collisions are assumed to occur randomly. The average number of collisions per unit length is given by $1/\lambda$. The probability of having no collisions in a path length of r is given by: $P_0(r)=e^{-r/\lambda}$

Average distance between collisions

$$\langle r\rangle_{col}=\frac{1}{\lambda}\int_0^\infty dr P_0(r) r=\lambda$$

Estimation of mean free path for hard spheres



Estimation of mean free path for hard spheres

Average relative speed between particles A and B:

$$\langle v_r\rangle_{AB}=\int d^3v_A F(v_A)\int d^3v_B F(v_B)\left|\mathbf{v}_A-\mathbf{v}_B\right|$$

Define relative velocity: $\mathbf{v}_r=\mathbf{v}_A-\mathbf{v}_B$ Relative mass: $\mu_{AB}=\frac{m_A m_B}{m_A+m_B}$

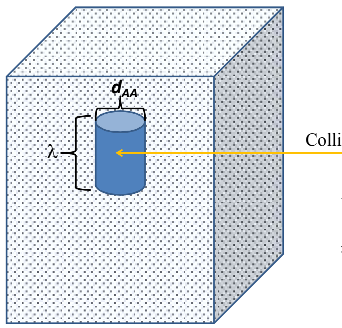
Center of mass velocity: $\mathbf{v}_{cm}=\frac{m_A\mathbf{v}_A+m_B\mathbf{v}_B}{m_A+m_B}$ Total mass: $M_{AB}=m_A+m_B$

$$\langle v_r\rangle_{AB}=\left(\frac{M_{AB}}{2\pi kT}\right)^{3/2}\left(\frac{\mu_{AB}}{2\pi kT}\right)^{3/2}\int d^3v_{cm}\int d^3v_r v_r \exp\left(-\frac{M_{AB}v_{cm}^2+\mu_{AB}v_r^2}{2kT}\right)$$

$$\langle v_r\rangle_{AB}=\left(\frac{8kT}{\pi\mu_{AB}}\right)^{1/2} \quad \text{for A} \neq B$$

$$\langle v_r\rangle_{AA}=\left(\frac{4kT}{\pi m_A}\right)^{1/2} \quad \text{for A} = B$$

When there is only one type of particle (A):

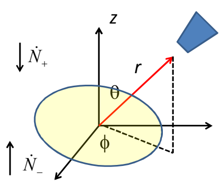


Collision volume: $\lambda \pi d_{AA}{}^2$

$$\lambda \pi d_{AA}{}^2 \approx \frac{V}{\sqrt{2}N_A}$$

$$\Rightarrow \lambda \approx \frac{1}{\sqrt{2}n_A \pi d_{AA}{}^2}$$

Self-diffusion



Assume that, in addition to geometric factors, the particle will reach the detector only if it does not have a collision.

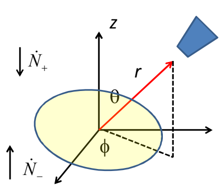
Recall the probability of not having a collision in a distance r: $P_0(r) = e^{-r/\lambda}$

$$\left(\dot{N}_+ - \dot{N}_-\right) = \frac{\langle v \rangle}{4\pi\lambda}\int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, n_T(z)\cos\theta \frac{e^{-r/\lambda}}{r^2}$$

Density of detected particles: $n_T(z) \approx n_T(0) + z\left(\frac{\partial n_z}{dz}\right)_0 + \ldots$

$$\Rightarrow \left(\dot{N}_+ - \dot{N}_-\right) = \frac{\langle v \rangle \lambda}{3}\left(\frac{\partial n_T}{\partial z}\right)_0$$

Self-diffusion



$$\left(\dot{N}_+ - \dot{N}_-\right) = \frac{\langle v \rangle \lambda}{3}\left(\frac{\partial n_T}{\partial z}\right)_0$$

$$\Downarrow \qquad \Downarrow$$

$$-J_D \qquad D$$

Self-diffusion equation:

$$\mathbf{J}_D(\mathbf{r},t) = -D\nabla_r n_T(\mathbf{r},t)$$

Continuity condition: $\frac{\partial n_T(\mathbf{r},t)}{\partial t} + \nabla_r \cdot \mathbf{J}_D(\mathbf{r}) = 0$

$$\rightarrow \frac{\partial n_T(\mathbf{r},t)}{\partial t} = D\nabla_r^2 n_T(\mathbf{r},t)$$

The Boltzmann Equation

(Additional reference: *Statistical Mechanics,* Kerson Huang)

Assume a dilute gas of N particles of mass m in a box of volume V. In order to justify a classical treatment:

$$\sqrt{\frac{\hbar^2}{2mkT}} << \left(\frac{V}{N}\right)^{1/3}$$

Define the distribution function $f(\mathbf{r},\mathbf{v},t)$:

$f(\mathbf{r},\mathbf{v},t)d^3rd^3v$ represents the number of particles in the 6 dimensional phase space about $\mathbf{r}$ and $\mathbf{v}$ at time t.

$$\int f(\mathbf{r},\mathbf{v},t)d^3rd^3v = N$$

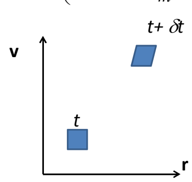
The Boltzmann equation – continued

In absence of collisions, the distribution of particles remains constant as

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{v}\delta t \qquad \mathbf{v} \rightarrow \mathbf{v}' = \mathbf{v} + \frac{\mathbf{F}}{m}\delta t \qquad t \rightarrow t' = t + \delta t$$

$$f(\mathbf{r},\mathbf{v},t)d^3rd^3v = f(\mathbf{r}',\mathbf{v}',t')d^3r'd^3v'$$

$$\Rightarrow f\left(\mathbf{r} + \mathbf{v}\delta t, \mathbf{v} + \frac{\mathbf{F}}{m}\delta t, t + \delta t\right) = f(\mathbf{r},\mathbf{v},t)$$



The Boltzmann equation – continued

In the presence of collisions there is a diffence between the two distributions such that:

$$f\left(\mathbf{r} + \mathbf{v}\delta t, \mathbf{v} + \frac{\mathbf{F}}{m}\delta t, t + \delta t\right) - f(\mathbf{r},\mathbf{v},t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \delta t$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r + \frac{\mathbf{F}}{m}\cdot\nabla_v\right) f(\mathbf{r},\mathbf{v},t) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

The Boltzmann Equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r + \frac{\mathbf{F}}{m}\cdot\nabla_v\right) f(\mathbf{r},\mathbf{v},t) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

The Boltzmann equation – continued

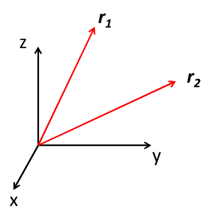
The Boltzmann Equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r + \frac{\mathbf{F}}{m}\cdot\nabla_v\right) f(\mathbf{r},\mathbf{v},t) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Linearized solution: $f(\mathbf{r},\mathbf{v},t) = f^0(v)\left(1 + h(\mathbf{r},\mathbf{v},t)\right)$

where $f^0(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}$

Digression on two particle scattering theory (see Appendix E)



$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{r}_{cm} = \mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Suppose that the interaction potential has the form $V(r)$:

$$H_{tot} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(r) = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(r)$$

Review of scattering analysis from classical mechanics class:

Scattering theory:

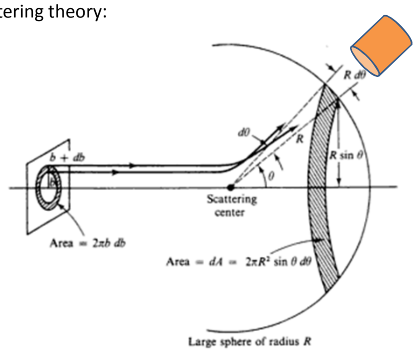
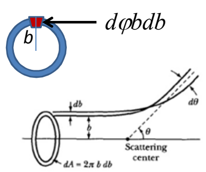


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{\text{Number of detected particles at }\theta\text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$= \text{Area of incident beam that is scattered into detector at angle } \theta$



$d\varphi b db$

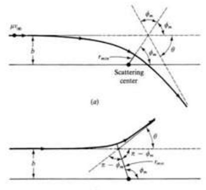
$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

How can we find $b(\theta)$?



Note that:

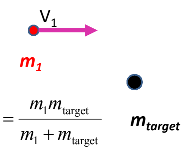
$\ell = \mu v_\infty b$

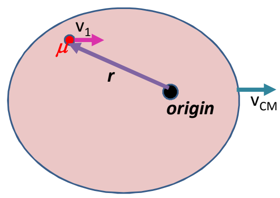
$\mu = \text{reduced mass}$

$v_\infty = \text{velocity at large separation}$

Note: The following is in the center of mass frame of reference.

In laboratory frame: In center-of-mass frame:





$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = |\mathbf{r} \times \mu \mathbf{v}_1|$$

Also note: We are assuming that the interaction between particle and target *V(r)* conserves energy and angular momentum.

Conservation of energy in the center of mass frame:

$$E = \frac{1}{2}\mu\left(\frac{d\mathbf{r}}{dt}\right)^2 + V(r)$$

$$= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

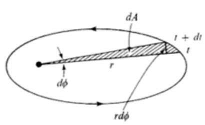
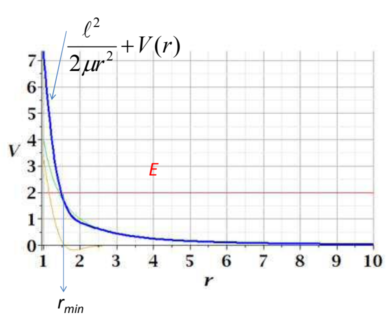
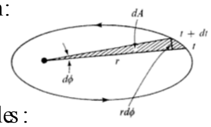


Figure 3.2 The areal velocity in a central field.

In center of mass reference frame:



Conservation of angular momentum:



$$\ell = \mu r^2 \left(\frac{d\phi}{dt} \right)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\phi)$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{\ell}{\mu r^2}$$

$$\Rightarrow E = \frac{1}{2}\mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2}\mu \left(\frac{dr}{d\phi}\frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\Rightarrow E = \frac{1}{2}\mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2}\mu \left(\frac{dr}{d\phi}\frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\phi) \Leftrightarrow \phi(r)$

$$\left(\frac{dr}{d\phi} \right)^2 = \left(\frac{2\mu r^4}{\ell^2} \right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)$$

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

Further simplification at large separation:

$$\ell = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

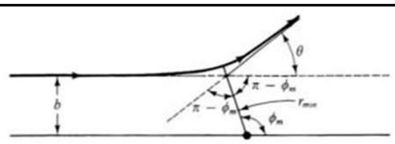
$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

When the dust clears :

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

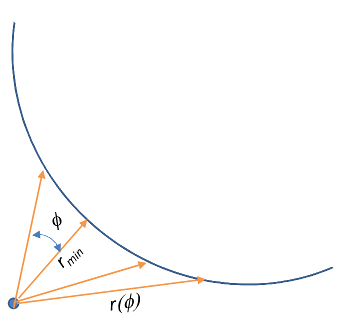
$$d\phi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \phi(b, E)$$



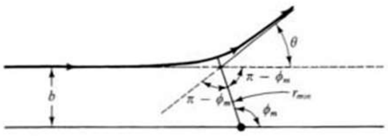
$$d\phi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\phi_m(b, E) = \int_{\infty}^{r_{\min}}\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)dr = b\int_0^{1/r_{\min}} \frac{du}{\sqrt{1 - b^2u^2 - V(1/u)/E}}$$



Evaluation of scattering expression:

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{2\pi\, b\, db}{2\pi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$



$$\theta+2(\pi-\phi_m)=\pi$$

$$\phi_m(b,E)=b\int_0^{1/r_{\min}}\frac{du}{\sqrt{1-b^2u^2-V(1/u)/E}}=\frac{\theta}{2}+\frac{\pi}{2}$$

Relationship between scattering angle θ and impact parameter *b* for interaction potential *V(r)*:

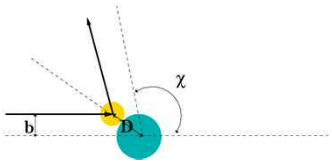
$$\theta=\pi-2b\int_{r_{\min}}^{\infty}dr\left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}}\right)$$

where :

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E}=0$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Hard sphere scattering



In this case:

$$b=D\cos\left(\frac{\chi}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right|=\frac{D^2}{4}$$

The results above were derived in the center of mass reference frame; relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before After

Center of mass reference frame:

Before After

Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1+m_2)\mathbf{R}_{CM}$$

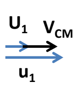
$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1+m_2)\dot{\mathbf{R}}_{CM}$$

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = (m_1+m_2)\mathbf{V}_{CM} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

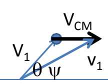
In our case :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2}$$

$$\mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM}$$



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

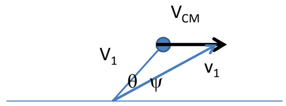
$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1+m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1+m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

For elastic scattering

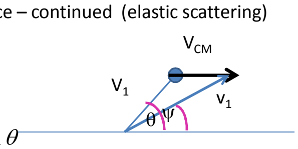
Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2 + \tfrac{1}{2}m_2U_2^2 + \tfrac{1}{2}(m_1+m_2)V_{CM}^2$$
$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \tfrac{1}{2}(m_1+m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1 + m_2\mathbf{U}_2 = 0 \qquad m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0$$
$$\mathbf{U}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM} \qquad \mathbf{U}_2 = -\mathbf{V}_{CM}$$
$$\Rightarrow |\mathbf{U}_1| = |\mathbf{V}_1| \quad \text{and} \quad |\mathbf{U}_2| = |\mathbf{V}_2| = |\mathbf{V}_{\mathbf{CM}}|$$
$$\text{Also note that: } m_1|\mathbf{U}_1| = m_2|\mathbf{U}_2|$$
$$\text{So that: } V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\text{Also: } \cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}\right|=\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

Using :

$$\cos\psi=\frac{\cos\theta+m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\theta+(m_1/m_2)^2}}$$

$$\left|\frac{d\cos\psi}{d\cos\theta}\right|=\frac{(m_1/m_2)\cos\theta+1}{\left(1+2(m_1/m_2)\cos\theta+(m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where : }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where : }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case : }\quad \tan\psi=\frac{\sin\theta}{\cos\theta+1}\quad\Rightarrow\psi=\frac{\theta}{2}$$

$$\text{note that}\quad 0\le\psi\le\frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Back to Boltzmann equation:
If we can assume that the collisions are due to binary interactions, such that particles 1 and 2 interact:

$$\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}'_2\} \quad \text{with cross section } \sigma(\Omega):$$

$$\left(\frac{\partial f(\mathbf{r}, \mathbf{v}_1, t)}{\partial t}\right)_{col} \approx \int dv_2{}^3 \int d\Omega \sigma(\Omega) \left|\mathbf{v}_2 - \mathbf{v}_1\right| \left(f'_2 f'_1 - f_2 f_1\right)$$

$$\text{where } \left(f'_2 f'_1 - f_2 f_1\right) \equiv \left(f(\mathbf{r}, \mathbf{v}'_1, t) f(\mathbf{r}, \mathbf{v}'_2, t) - f(\mathbf{r}, \mathbf{v}_1, t) f(\mathbf{r}, \mathbf{v}_2, t)\right)$$