PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 4:

Reading: Chapter 1 & 2 in JDJ

Electrostatic potentials

1. **One, two, and three dimensions (Cartesian coordinates)**
2. **Mean value theorem for the electrostatic potential**

PHY 712 Electrodynamics

MWF 9-9:50 AM | OPL 103 | http://www.wfu.edu/~natalie/s15phy712/

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Course schedule for Spring 2015

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	Assign.	Due date
1	Mon: 01/12/2015	Chap. 1	Introduction, units and Poisson equation	#1	01/23/2015
2	Wed: 01/14/2015	Chap. 1	Electrostatic energy calculations	#2	01/23/2015
	Fri: 01/16/2015	No class	NAWH out of town		
	Mon: 01/19/2015	No class	MLK Holiday		
3	Wed: 01/21/2015	Chap. 1	Poisson equation and Green's theorm	#3	01/23/2015
4	Fri: 01/23/2015	Chap. 1 & 2	Green's functions in Cartesian coordinates	#4	01/25/2015

Poisson Equation

$$\nabla^2 \Phi_P(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Solution to Poisson equation using Green's function $G(\mathbf{r},\mathbf{r}')$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r' \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r' \left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

Poisson equation for one-dimensional system

$$\frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

Example solution:

$$\Phi_P(x) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} G(x,x')\rho(x')dx' + C_1 + C_2 x$$

where $G(x,x') = 4\pi x_<$ where $x_<$ is the smaller of x and x';
C_1 and C_2 are constants.

Check:

$$\Phi_P(x) = \frac{1}{\varepsilon_0}\left\{\int_{-\infty}^{x} x'\rho(x')dx' + x\int_{x}^{\infty}\rho(x')dx'\right\} + C_1 + C_2 x$$

$$\frac{d\Phi_P(x)}{dx} = \frac{1}{\varepsilon_0}\int_{x}^{\infty}\rho(x')dx' + C_2 \quad \Rightarrow \frac{d^2\Phi_P(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

General procedure for constructing Green's function for one-dimensional system using 2 independent solutions of the homogeneous equations

Consider two independent solutions to the homogeneous equation
$\nabla^2\phi_i(x) = 0$
where $i = 1$ or 2. Let

$$G(x,x') = \frac{4\pi}{W}\phi_1(x_<)\phi_2(x_>).$$

This notation means that $x_<$ should be taken as the smaller of x and x' and $x_>$ should be taken as the larger.

"Wronskian": $W \equiv \frac{d\phi_1(x)}{dx}\phi_2(x) - \phi_1(x)\frac{d\phi_2(x)}{dx}.$

Beautiful method; but only works in one dimension.

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \le x \le x_2$ such that

$$\int_{x_1}^{x_2} u_n(x)u_m(x)\,dx = \delta_{nm}.$$

We can show that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x)u_n(x') = \delta(x-x').$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2}G(x,x') = -4\pi\delta(x-x').$$

Orthogonal function expansion -- continued

Suppose the orthogonal functions satisfy an eigenvalue equation:

$$\frac{d^2}{dx^2}u_n(x) = -\alpha_n u_n(x)$$

where the functions $u_n(x)$ also satisfy the appropriate boundary conditions, then we can construct the Green's function:

$$G(x,x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}.$$

Check:

$$\frac{d^2}{dx^2}G(x,x') = 4\pi \sum_n \frac{\left(-\alpha_n u_n(x)\right)u_n(x')}{\alpha_n} = -4\pi \sum_n u_n(x)u_n(x')$$
$$= -4\pi\delta(x-x')$$

Example

For example, consider the previous example in the interval $-a \le x \le a$:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

We want to solve the Poisson equation with boundary condition $d\Phi(-a)/dx = 0$ and $d\Phi(a)/dx = 0$. We may choose $u_n(x) = \sqrt{\frac{1}{a}}\sin\left(\frac{[2n+1]\pi x}{2a}\right)$ and the corresponding Green's function

$$G(x,x') = \frac{4\pi}{a}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)\sin\left(\frac{[2n+1]\pi x'}{2a}\right)}{\left(\frac{[2n+1]\pi}{2a}\right)^2}.$$

Example -- continued

This form of the one-dimensional Green's function only allows us to find a solution to the Poisson equation within the interval $-a \le x \le a$ from

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a} dx'\, G(x,x')\rho(x') + C_1$$

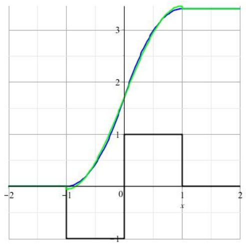
$$\Rightarrow \Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right),$$

choosing C_1 so that $\Phi(-a) = 0$.

Exact result: $$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{2\varepsilon_0}(x+a)^2 & \text{for } -a < x < 0 \\ -\frac{\rho_0}{2\varepsilon_0}(x-a)^2 + \frac{\rho_0 a^2}{\varepsilon_0} & \text{for } 0 < x < a \\ \frac{\rho_0}{\varepsilon_0}a^2 & \text{for } x > a \end{cases}$$

Example -- continued

$$\Phi(x) = \frac{\rho_0 a^2}{\epsilon_0}\left(16\sum_{n=0}^{\infty}\frac{\sin\left(\frac{[2n+1]\pi x}{2a}\right)}{([2n+1]\pi)^3} + \frac{1}{2}\right)$$



Orthogonal function expansions in 2 and 3 dimensions

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r})/\epsilon_0.$$

Let $\{u_n(x)\}, \{v_n(y)\}, \{w_n(z)\}$ denote complete orthogonal function sets in the x, y, and z dimensions, respectively. The Green's function construction becomes:

$$G(x,x',y,y',z,z') = 4\pi\sum_{lmn}\frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n},$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x),\ \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y),\ \text{and}\ \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

(See Eq. 3.167 in Jackson for example.)

Combined orthogonal function expansion and homogeneous solution construction of Green's function in 2 and 3 dimensions.

An alternative method of finding Green's functions for second order ordinary differential equations is based on a product of two independent solutions of the homogeneous equation, $\phi_1(x)$ and $\phi_2(x)$:

$$G(x,x') = K\phi_1(x_<)\phi_2(x_>),\ \text{where}\ K \equiv \frac{4\pi}{\frac{d\phi_1}{dx}\phi_2 - \phi_1\frac{d\phi_2}{dx}},$$

where $x_<$ denotes the smaller of x and x'.

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of Jackson.

Green's function construction -- continued

For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')g_n(y,y').$$

The y dependence of this equation will have the required behavior, if we choose: $\left[-\alpha_n + \frac{\partial^2}{\partial y^2}\right]g_n(y,y') = -4\pi\delta(y-y'),$

which in turn can be expressed in terms of the two independent solutions $v_{n_1}(y)$ and $v_{n_2}(y)$ of the homogeneous equation:

$$\left[\frac{d^2}{dy^2} - \alpha_n\right]v_{n_i}(y) = 0,$$

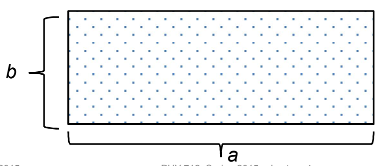
and the Wronskian constant: $K_n \equiv \dfrac{4\pi}{\dfrac{dv_{n_1}}{dy}v_{n_2} - v_{n_1}\dfrac{dv_{n_2}}{dy}}.$

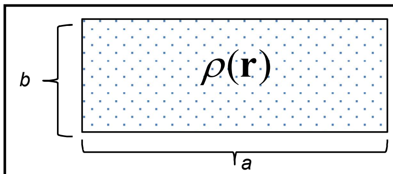
Green's function construction -- continued

$$G(x,x',y,y') = \sum_n u_n(x)u_n(x')K_n v_{n_1}(y_<)v_{n_2}(y_>).$$

For example, a Green's function for a two-dimensional rectangular system with $0 \le x \le a$ and $0 \le y \le b$, which vanishes on the rectangular boundaries

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$





$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') +$$

$$\frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

$$G(x,x',y,y') = 8\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}.$$

A useful theorem for electrostatics
The mean value theorem (Problem 1.10 in Jackson)

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots . \quad (1)$$

According to the premise of the theorem, we want to integrate both sides of the equation 1 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \quad (2)$$

Mean value theorem – continued

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u}\cdot\nabla = 0,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^2 = \frac{4\pi R^4}{3}\nabla^2,$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^3 = 0,$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u}\cdot\nabla)^4 = \frac{4\pi R^6}{5}\nabla^4.$$

Since $\nabla^2\Phi(\mathbf{r}) = 0$, the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r}+\mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}),$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\ R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r}+\mathbf{u}) \equiv \frac{1}{4\pi R^2} \int_{\text{sphere}} dS_u \Phi(\mathbf{r}+\mathbf{u}).$$

Since this result is independent of the radius R, we see that we have the theorem.