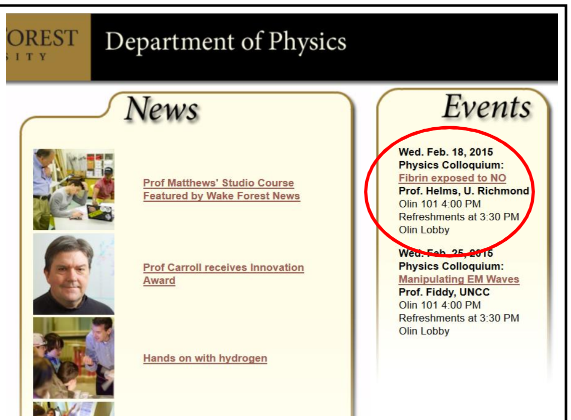
PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 15:

Finish reading Chapter 6

1. Some details of Liénard-Wiechert results
2. Energy density and flux associated with electromagnetic fields
3. Time harmonic fields

7	Fri: 01/30/2015	Chap. 3	Cylindrical and spherical geometries	#7	02/02/2015
8	Mon: 02/02/2015	Chap. 4	Multipole analysis	#8	02/04/2015
9	Wed: 02/04/2015	Chap. 4	Dipoles and dielectrics	#9	02/06/2015
10	Fri: 02/06/2015	Chap. 4	Dipoles and dielectrics	#10	02/09/2015
11	Mon: 02/09/2015	Chap. 5	Magnetostatics	#11	02/11/2015
12	Wed: 02/11/2015	Chap. 5	Magnetostatics	#12	02/13/2015
13	Fri: 02/13/2015	Chap. 5	Magnetostatics	#13	02/16/2015
14	Mon: 02/16/2015	Chap. 6	Maxwell's equations	#13	02/18/2015
15	Wed: 02/18/2015	Chap. 6	Electromagnetic energy and force	#15	02/20/2015
16	Fri: 02/20/2015	Chap. 7	Electromagnetic plane waves	#16	02/23/2015
17	Mon: 02/23/2015	Chap. 7	Dielectric media	#17	02/25/2015
18	Wed: 02/25/2015	Chap. 7	Complex dielectrics	#18	02/27/2015
19	Fri: 02/27/2015	Chap. 8	Wave guides		
	Mon. 03/02/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/04/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/06/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/09/2015	*Spring Break*			
	Wed. 03/11/2015	*Spring Break*			
	Fri. 03/13/2015	*Spring Break*			
20	Fri: 02/27/2015	Chap. 8	Wave guides		



WFU Physics Colloquium

TITLE: Structural and Mechanical Properties of Fibrin Exposed to Nitric Oxide

SPEAKER: Dr. Christine Helms,

Department of Physics
University of Richmond

TIME: Wednesday February 18, 2015 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Fibrin fibers are a major component of blood clots, which perform the mechanical and structural task of stemming the flow of blood. Therefore mechanical and structural properties of clots and their constituent fibrin fibers are important for understanding and describing clot function. Studies of blood clots have related myocardial infarction, coronary artery disease and diabetes mellitus to the mechanical properties and the structure of blood clots. Additionally, environmental factors such as pH and ionic strength have been shown to affect clot properties. Nitric oxide (NO) a small biological molecule produced by endothelial cells and linked to atherosclerosis, diabetes and hypertension may affect fibrin fiber

Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz ***The Classical Theory of Fields***, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $R_q(t)$.

Charge density: $\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r}-\mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\int d^3r'dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\big)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\int d^3r'dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\big(t'-(t-|\mathbf{r}-\mathbf{r}'|/c)\big).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t',

$$\int_{-\infty}^{\infty} dt'\, f(t')\delta\big(t'-(t-|\mathbf{r}-\mathbf{R}_q(t')|/c)\big) = \frac{f(t_r)}{1-\frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the ``retarded time'' is defined to be

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{R}_q(t_r)|}{c}.$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r), \qquad t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 7

Solution of Maxwell's equations in the Lorentz gauge -- continued

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

The trick of evaluating these derivatives is that the retarded time t_r depends on position **r** and on itself. We can show the following results using the shorthand notation:

$$\nabla t_r = -\frac{\mathbf{R}}{c\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)} \quad \text{and} \quad \frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)}.$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 8

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$-\nabla\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\mathbf{R}\left(1 - \frac{v^2}{c^2}\right) - \frac{\mathbf{v}}{c}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right) + \mathbf{R}\frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right],$$

$$-\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\frac{\mathbf{v}R}{c}\left(\frac{v^2}{c^2} - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}}\cdot R}{c^2}\right) - \frac{\dot{\mathbf{v}}R}{c^2}\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)\right].$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R} \times \left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right) \times \frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right].$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right] = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{cR}$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 9

Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field :

$$\frac{dW}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \ \mathbf{E} \cdot \mathbf{J}$$

Expressing source current in terms of fields it produces :

$$\frac{dW}{dt} = \int d^3r \ \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dW}{dt} = \int d^3r \ \mathbf{E} \cdot \mathbf{J} = \int d^3r \ \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$= -\int d^3r \left(\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)$$

Let $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \ \mathbf{E} \cdot \mathbf{J}$$

Electromagnetic energy density : $u \equiv \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$

$$E_{field} \equiv \int d^3r \ u(\mathbf{r},t)$$

Poynting vector : $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

From the previous energy analysis : $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \ \nabla \cdot \mathbf{S}(\mathbf{r},t) = -\oint d^2r \ \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r},t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \;\; (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})$$

Follows by analogy with Lorentz force:

$$F = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \;\; (\mathbf{E}\times\mathbf{B})$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \; \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j + c^2B_iB_j - \delta_{ij}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow$ $\tilde{\mathbf{E}}(\mathbf{r},\omega) = \tilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \tilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Equations	in time domain	in frequency domain
Coulomb's law:	$\nabla\cdot\mathbf{D} = \rho_{free}$	$\nabla\cdot\tilde{\mathbf{D}} = \tilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla\times\tilde{\mathbf{H}} + i\omega\tilde{\mathbf{D}} = \tilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\times\tilde{\mathbf{E}} - i\omega\tilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\tilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right) \times \left(\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega) + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\langle \mathbf{S}(\mathbf{r},t) \rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

Summary and review

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

For linear isotropic media -- $\mathbf{D} = \varepsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$

and no sources:

Coulomb's law: $\nabla \cdot \mathbf{E} = 0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law}: \quad \nabla \cdot \mathbf{E} = 0$$

$$\text{Ampere-Maxwell's law}: \quad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\text{Faraday's law}: \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles}: \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 19

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 20

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$$

Note: *ε, μ, n, k* can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

$$\text{from Faraday's law}: \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

$$\text{also note}: \quad \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 \quad \text{and} \quad \hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$$

02/18/2015 PHY 712 Spring 2015 -- Lecture 15 21

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg} = \frac{1}{2}\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\times\frac{1}{\mu}\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Analysis of Maxwell's equations without sources -- continued:

Transverse Electric and Magnetic (TEM) waves

Summary of plane electromagnetic waves:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \quad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{4}\Re\left(\varepsilon\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right) +$$

$$\frac{1}{4}\Re\left(\frac{1}{\mu}\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\cdot\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)^*\right)$$

$$= \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

More general result (for homework)

1. Suppose that an electromagnetic wave of pure (real) frequency ω is traveling along the z-axis of a wave guide having a square cross section with side dimension a composed of a medium having a real permittivity constant ε and a real permeability constant μ. Suppose that the wave is known to have the form:

$$\mathbf{E}(\mathbf{r},t) = \Re\left\{H_0 e^{ikz-i\omega t}\left[(i\mu\omega)\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{y}}\right]\right\}$$

$$\mathbf{H}(\mathbf{r},t) = \Re\left\{H_0\ \mathrm{e}^{ikz-i\omega t}\left[-ik\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\hat{\mathbf{x}} + \cos\left(\frac{\pi x}{a}\right)\hat{\mathbf{z}}\right]\right\}.$$

Here H_0 denotes a real amplitude, and the parameter k is assumed to be real and equal to

$$k \equiv \sqrt{\mu\epsilon\omega^2 - \left(\frac{\pi}{a}\right)^2}, \quad \text{for} \quad \mu\epsilon\omega^2 > \left(\frac{\pi}{a}\right)^2$$

a. Show that this wave satisfies the sourceless Maxwell's equations.
b. Find the form of the time-averaged Poynting vector

$$\langle\mathbf{S}\rangle_{avg} \equiv \frac{1}{2}\Re\left\{\mathbf{E}(\mathbf{r},t)\times\mathbf{H}^*(\mathbf{r},t)\right\}$$

for this electromagnetic wave.