PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 20:

1. Review of Mid-term Exam
2. Electromagnetic waves within waveguides

10	Fri: 02/06/2015	Chap. 4	Dipoles and dielectrics	#10	02/09/2015
11	Mon: 02/09/2015	Chap. 5	Magnetostatics	#11	02/11/2015
12	Wed: 02/11/2015	Chap. 5	Magnetostatics	#12	02/13/2015
13	Fri: 02/13/2015	Chap. 5	Magnetostatics	#13	02/16/2015
14	Mon: 02/16/2015	Chap. 6	Maxwell's equations	#14	02/18/2015
15	Wed: 02/18/2015	Chap. 6	Electromagnetic energy and force	#15	02/20/2015
16	Fri: 02/20/2015	Chap. 7	Electromagnetic plane waves	#16	02/23/2015
17	Mon: 02/23/2015	Chap. 7	Dielectric media	#17	02/25/2015
18	Wed: 02/25/2015	Chap. 7	Complex dielectrics	#18	02/27/2015
19	Fri: 02/27/2015	Chap. 1-7	Review -- Take home exam distributed		
	Mon. 03/02/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Wed. 03/04/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Fri. 03/06/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/09/2015	*Spring Break*			
	Wed. 03/11/2015	*Spring Break*			
	Fri. 03/13/2015	*Spring Break*			
20	Mon: 03/16/2015	Chap. 8	Review Exam; Wave guides	#19	03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20	03/20/2015

Review of Mid-Term Exam Problems

1. Consider the following one-dimensional charge distribution in vacuum:

$$\rho(x) = \begin{cases} 0 & x < -a/2 \\ \rho_0 \cos(\pi x/a) & -a/2 \le x \le 3a/2 \\ 0 & x > 3a/2 \end{cases}$$

where ρ_0 is a constant having the units of charge per unit volume and a is a length constant. It is assumed that the charge density and all related fields are uniform in the y and z directions.

(a) Find the electrostatic potential as a function of x for all $-\infty \le x \le \infty$ due to this charge distribution, assuming the boundary conditions

$$\left.\frac{d\Phi(x)}{dx}\right]_{-\infty} = 0 \quad \text{and} \quad \left.\frac{d\Phi(x)}{dx}\right]_{\infty} = 0.$$

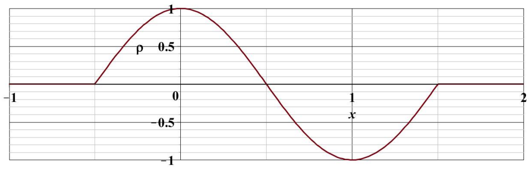
(b) Determine the value of $\Phi(x=\infty) - \Phi(x=-\infty)$.

(c) Determine the corresponding x-component of the electrostatic field as a function of x:

$$E_x(x) = -\frac{d\Phi(x)}{dx}.$$

(d) Find the corresponding electrostatic energy W and electrostatic energy density $w(x)$ discussed in Eqs. 1.54 and 1.55 of your textbook.

Use Maple or another program to generate plots of your results.



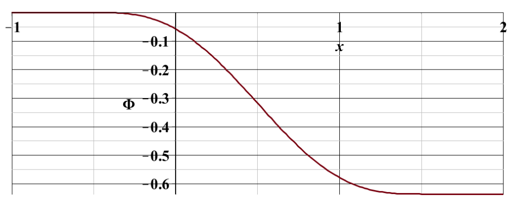
$> f := (x, a) \rightarrow piecewise\left(x < -\frac{a}{2}, 0, x > -\frac{a}{2} \textbf{ and } x < \frac{3 \cdot a}{2}, \cos\left(\frac{\text{Pi} \cdot x}{a}\right), x > \frac{3 \cdot a}{2}, 0\right);$

$$f := (x, a) \rightarrow piecewise\left(x < -\frac{1}{2}a, 0, -\frac{1}{2}a < x \textbf{ and } x < \frac{3}{2}a, \cos\left(\frac{\pi x}{a}\right), \frac{3}{2}a < x, 0\right)$$

Potential is determined up to a constant:

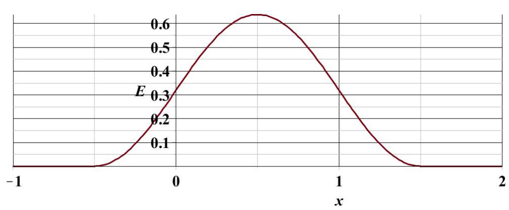
$$\Phi(x) = \frac{1}{\epsilon_0}\left(\int_{-\infty}^{x} du\ u\rho(u) + x\int_{x}^{\infty} \rho(u)du\right)$$

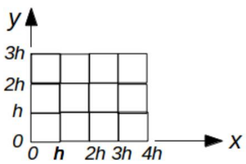
$$\Phi(x) = \begin{cases} 0 & x < -a/2 \\ \dfrac{a^2}{\epsilon_0 \pi^2}\left(\cos\left(\dfrac{\pi x}{a}\right) - \pi\left(\dfrac{x}{a} + \dfrac{1}{2}\right)\right) & -a/2 < x < 3a/2 \\ -\dfrac{2a^2}{\epsilon_0 \pi} & x > 3a/2 \end{cases}$$



$$E(x) = -\frac{d\Phi(x)}{dx}$$

$$E(x) = \begin{cases} 0 & x < -a/2 \\ \dfrac{a}{\epsilon_0 \pi}\left(\sin\left(\dfrac{\pi x}{a}\right) + 1\right) & -a/2 < x < 3a/2 \\ 0 & x > 3a/2 \end{cases}$$





2.

Consider the area shown in the diagram above where $0 \le x \le 4h$ and $0 \le y \le 3h$. A charge density within this area is given by

$$\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{4h}\right) \sin\left(\frac{\pi y}{3h}\right).$$

In this problem, you will find the corresponding electrostatic potential $\Phi(x,y,)$ with the boundary conditions $\Phi(0,y) = \Phi(x,0) = \Phi(4h,y) = \Phi(x,3h) = 0$, using 3 different methods.

(a) First find analytic form for $\Phi(x,y)$ using Green's functions or other methods of your choice.

(b) Now find the numerical approximation to $\Phi(x,y)$ using the finite difference method discussed in class and in your text book. Compare your numerical results with the analytic results.

(c) Finally find the numerical approximation to $\Phi(x,y)$ using the finite element method discussed in class and in your text book. Compare your numerical results with the analytic results.

Poisson equation:

$$\nabla^2\Phi(x,y) = -\frac{\rho(x,y)}{\epsilon_0} \quad \text{where } \rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right)$$

Solution:

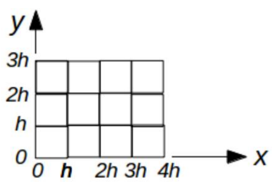
$$\Phi(x,y) = \frac{\rho_0}{\epsilon_0\left(\left(\frac{\pi}{4h}\right)^2 + \left(\frac{\pi}{3h}\right)^2\right)} \sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right)$$

Exact values on grid points:

$\Phi(h,h) = \Phi(h,2h) =$

$\Phi(3h,h) = \Phi(3h,2h) = \frac{\rho_0 h^2}{\epsilon_0} 0.3573866880$

$\Phi(2h,h) = \Phi(2h,2h) = \frac{\rho_0 h^2}{\epsilon_0} 0.5054211012$



For numerical solutions, refer to Lecture 5:

Finite difference:

$$S_A \equiv \Phi(x+h,y) + \Phi(x-h,y) + \Phi(x,y+h) + \Phi(x,y-h)$$

$$S_B \equiv \Phi(x+h,y+h) + \Phi(x-h,y+h) + \Phi(x+h,y-h) + \Phi(x-h,y-h)$$

$$\Phi(x,y) - \frac{1}{5}S_A - \frac{1}{20}S_B = \frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y).$$

Linear algebra problem of the form: $Mx = b$ where, if no symmetry is used, M is a 6×6 matrix:

$$\begin{bmatrix} 1 & -\frac{1}{5} & 0 & -\frac{1}{5} & -\frac{1}{20} & 0 \\ -\frac{1}{5} & 1 & -\frac{1}{5} & -\frac{1}{20} & -\frac{1}{5} & -\frac{1}{20} \\ 0 & -\frac{1}{5} & 1 & 0 & -\frac{1}{20} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{20} & 0 & 1 & -\frac{1}{5} & 0 \\ -\frac{1}{20} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 1 & -\frac{1}{20} \\ 0 & -\frac{1}{20} & -\frac{1}{5} & 0 & -\frac{1}{5} & 1 \end{bmatrix}$$

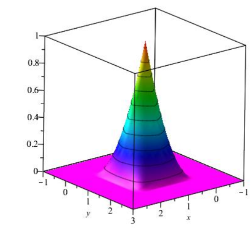
Finite element method:

$$\Phi(x,y)=\sum_{ij}\psi_{ij}\phi_{ij}(x,y), \qquad \sum_{ij}M_{kl,ij}\psi_{ij}=G_{kl},$$

$$M_{kl,ij}=\int dx\int dy \nabla\phi_{kl}(x,y)\cdot\nabla\phi_{ij}(x,y)$$

$$G_{kl}=\frac{1}{\epsilon_0}\int dx\int dy \phi_{kl}(x,y)\rho(x,y)$$

$\phi_{11}(x,y)$



Convenient choice of finite elements --

$$\phi_{ij}(x,y)\equiv\mathcal{X}_i(x)\mathcal{Y}_j(y),$$

$$\mathcal{X}_i(x)\equiv\begin{cases}\left(1-\frac{|x-x_i|}{h}\right) & \text{for } x_i-h\leq x\leq x_i+h\\ 0 & \text{otherwise}\end{cases},$$

$$M_{kl,ij}=\begin{cases}\frac{8}{3} & \text{for } k=i \text{ and } l=j\\ -\frac{1}{3} & \text{for } k-i=\pm1 \text{ and/or } l-j=\pm1\\ 0 & \text{otherwise}\end{cases}$$

Form of *M* for without using symmetry:

$$\begin{bmatrix}\frac{8}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0\\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\\ 0 & -\frac{1}{3} & \frac{8}{3} & 0 & -\frac{1}{3} & -\frac{1}{3}\\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & 0\\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3}\\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{8}{3}\end{bmatrix}$$

3. Consider a 3-dimensional charge density expressed in spherical polar coordinates

$$\rho(r,\theta,\phi)=Q_0 e^{-(r/a)^2}+Q_1\frac{r}{b}e^{-(r/b)^2}\cos\theta,$$

where Q_0 and Q_1 are constants having the units of charge per unit volume and a and b are length parameters.

(a) Find an analytic expression for the electrostatic potential as a function of position.

(b) Analyze the potential for $r\to 0$ and describe its qualitative features.

(c) Analyze the potential for $r\to\infty$ and describe its qualitative features. In particular, can you write the potential in the form

$$\Phi(\mathbf{r})\overset{r\to\infty}{=}\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}+\dots\right),$$

and if so, explain.

From Lecture 8:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

General form of electrostatic potential with boundary value $r\rightarrow\infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r})=\sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow\Phi(\mathbf{r})=\frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r')+r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

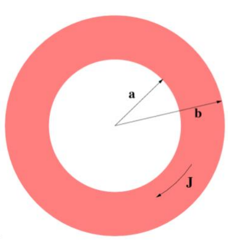
For our case:

$$\rho(r,\theta,\phi)=Q_0e^{-(r/a)^2}+Q_1\frac{r}{b}e^{-(r/b)^2}\cos\theta$$

$$\rho_{00}(r)=\sqrt{4\pi}Q_0e^{-(r/a)^2}$$

$$\rho_{10}(r)=\sqrt{\frac{4\pi}{3}}Q_1\frac{r}{b}e^{-(r/b)^2}$$

$$\Phi(\mathbf{r})=\frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r')+r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$



4.

The figure above shows the cross section of a magnetostatic solenoid which is uniform in the $\hat{\mathbf{z}}$ direction (perpendicular to the page). The current flows in the azimuthal $\hat{\phi}$ direction; specifically the current density is given in cylindrical coordinates by:

$$\mathbf{J}=\begin{cases}J_0\hat{\phi} & a\leq\rho\leq b\\ 0 & \text{otherwise.}\end{cases}\quad(1)$$

Here J_0 is a constant, a and b denote the inner and outer diameters of the cylinder, respectively, and $\hat{\phi}=-\sin(\phi)\hat{\mathbf{x}}+\cos(\phi)\hat{\mathbf{y}}$.

(a) Show that the vector potential **A** for this system can be written as

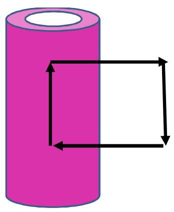
$$\mathbf{A} = f(\rho)\hat{\phi}, \quad (2)$$

where the scalar function $f(\rho)$ satisfies the equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\right] f(\rho) = \begin{cases} -\mu_0 J_0 & a \leq \rho \leq b \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

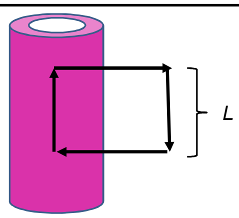
(b) Find the function $f(\rho)$ in the three regions: $0 \leq \rho \leq a$, $a \leq \rho \leq b$, and $\rho \geq b$.

(c) Find the **B** field in the three regions. Check to make sure that your answer is consistent with what you know about solenoids. (Hint: $\mathbf{B} \equiv \mathbf{0}$ outside the solenoid.)



Integral form of Ampere's law:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$$

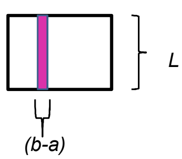


Integral form of Ampere's law:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$$

$$B_{in} L = \mu_0 J_0 (b-a) L$$

$$B_{in} = \mu_0 J_0 (b-a)$$



Physics of wave guides --

Fields near the surface on an ideal conductor

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\big|_S = 0$$

Suppose for an isotropic medium: $\mathbf{D}=\varepsilon_b\mathbf{E}$ $\quad \mathbf{J}=\sigma\mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\cdot\mathbf{H}=0$$

$$\nabla\times\mathbf{E}=-\mu\frac{\partial\mathbf{H}}{\partial t} \qquad \nabla\times\mathbf{H} = \sigma\mathbf{E} + \varepsilon_b\frac{\partial\mathbf{E}}{\partial t}$$

$$\left(\nabla^2-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon_b\frac{\partial^2}{\partial t^2}\right)\mathbf{F}=0 \qquad \mathbf{F}=\mathbf{E},\mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right) \qquad \text{where } \mathbf{k}=(n_R+in_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c}n_R=\omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}+1\right)^{1/2}$$

$$\frac{\omega}{c}n_I=\omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon_b}\right)^2}-1\right)^{1/2}$$

$$\text{For } \frac{\sigma}{\omega}>>1 \qquad \frac{\omega}{c}n_R\approx\frac{\omega}{c}n_I\approx\sqrt{\frac{\mu\sigma\omega}{2}}\equiv\frac{1}{\delta}$$

$$\Rightarrow\mathbf{E}(\mathbf{r},t)\approx e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta-i\omega t}\right)$$

$$\Rightarrow\mathbf{H}(\mathbf{r},t)\approx\Re\left(\frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)\right)\approx\Re\left(\frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)\right)$$

20

Boundary values for ideal conductor

Inside the conductor:

$$\mathbf{H}(\mathbf{r},t)=e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta-i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t)=\delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface.

Ideal conductor boundary conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}\big|_S = 0 \qquad \hat{\mathbf{n}}\cdot\mathbf{H}\big|_S = 0$$